Problem 16

Consider the initial value problem

$$y'' + y = f_k(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where $f_k(t) = [u_{4-k}(t) - u_{4+k}(t)]/2k$ with $0 < k \leq 1$.

(a) Find the solution $y = \phi(t, k)$ of the initial value problem.

(b) Calculate $\lim_{k \to 0^+} \phi(t, k)$ from the solution found in part (a).

(c) Observe that $\lim_{k \to 0^+} f_k(t) = \delta(t - 4)$. Find the solution $\phi_0(t)$ of the given initial value problem with $f_k(t)$ replaced by $\delta(t - 4)$. Is it true that $\phi_0(t) = \lim_{k \to 0^+} \phi(t, k)$?

(d) Plot $\phi(t, 1/2)$, $\phi(t, 1/4)$, and $\phi_0(t)$ on the same axes. Describe the relation between $\phi(t, k)$ and $\phi_0(t)$.

Solution

Part (a)

Because the ODE is linear, the Laplace transform can be used to solve it. The Laplace transform of a function $y(t)$ is defined to be

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Substitute the function for $f_k(t)$ and take the Laplace transform of both sides of the ODE.

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\left\{\frac{1}{2k}[u_{4-k}(t) - u_{4+k}(t)]\right\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \frac{1}{2k}\mathcal{L}\{u_{4-k}(t)\} - \frac{1}{2k}\mathcal{L}\{u_{4+k}(t)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + Y(s) = \frac{1}{2k}\int_0^\infty e^{-st} u_{4-k}(t)\, dt - \frac{1}{2k}\int_0^\infty e^{-st} u_{4+k}(t)\, dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$$[s^2Y(s)] + Y(s) = \frac{1}{2k}\int_{4-k}^\infty e^{-st}\, dt - \frac{1}{2k}\int_{4+k}^\infty e^{-st}\, dt$$

Evaluate the integrals.

$$(s^2+1)Y(s) = \frac{1}{2k}\left[\frac{e^{-s(4-k)}}{s}\right] - \frac{1}{2k}\left[\frac{e^{-s(4+k)}}{s}\right]$$

Solve for $Y(s)$ and use partial fraction decomposition to write it in terms of known transforms.

$$\begin{aligned} Y(s) &= \frac{1}{2k}\left[\frac{1}{s(s^2+1)}\right]e^{-s(4-k)} - \frac{1}{2k}\left[\frac{1}{s(s^2+1)}\right]e^{-s(4+k)} \\ &= \frac{1}{2k}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-s(4-k)} - \frac{1}{2k}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-s(4+k)} \end{aligned}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ \phi(t,k) &= \mathcal{L}^{-1}\left\{\frac{1}{2k}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-s(4-k)} - \frac{1}{2k}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-s(4+k)}\right\} \\ &= \frac{1}{2k}\mathcal{L}^{-1}\left\{\left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-s(4-k)}\right\} - \frac{1}{2k}\mathcal{L}^{-1}\left\{\left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-s(4+k)}\right\} \\ &= \frac{1}{2k}\{1-\cos[t-(4-k)]\}H[t-(4-k)] - \frac{1}{2k}\{1-\cos[t-(4+k)]\}H[t-(4+k)] \\ &= \frac{\{1-\cos[t-(4-k)]\}H[t-(4-k)] - \{1-\cos[t-(4+k)]\}H[t-(4+k)]}{2k} \end{aligned}$$

Part (b)

Here we will calculate the limit of $y(t)$ as $k \to 0$ from the right.

$$\begin{aligned} \lim_{k\to 0^+} y(t) &= \lim_{k\to 0^+} \frac{\{1-\cos[t-(4-k)]\}H[t-(4-k)] - \{1-\cos[t-(4+k)]\}H[t-(4+k)]}{2k} \\ &= \lim_{k\to 0^+}\left\{\frac{1-\cos[t-(4-k)]}{2k}H[t-(4-k)]\right\} - \lim_{k\to 0^+}\left\{\frac{1-\cos[t-(4+k)]}{2k}H[t-(4+k)]\right\} \\ &= \left\{\lim_{k\to 0^+}\frac{1-\cos[t-(4-k)]}{2k}\right\}\left\{\lim_{k\to 0^+} H[t-(4-k)]\right\} \\ &\qquad - \left\{\lim_{k\to 0^+}\frac{1-\cos[t-(4+k)]}{2k}\right\}\left\{\lim_{k\to 0^+} H[t-(4+k)]\right\} \\ &= \left\{\lim_{k\to 0^+}\frac{1-\cos[t-(4-k)]}{2k}\right\}H(t-4) - \left\{\lim_{k\to 0^+}\frac{1-\cos[t-(4+k)]}{2k}\right\}H(t-4) \\ &= \left\{\lim_{k\to 0^+}\frac{1-\cos[t-(4-k)]}{2k} - \lim_{k\to 0^+}\frac{1-\cos[t-(4+k)]}{2k}\right\}H(t-4) \\ &= \lim_{k\to 0^+}\frac{-\cos[t-(4-k)]+\cos[t-(4+k)]}{2k}H(t-4) \\ &\overset{\frac{0}{0}}{\underset{\text{H}}{=}} \lim_{k\to 0^+}\frac{\sin[t-(4-k)]-\sin[t-(4+k)](-1)}{2}H(t-4) \\ &= \sin(t-4)H(t-4) \end{aligned}$$

Part (c)

Calculate the limit of $f_k(t)$, noting that $H'(k) = \delta(k)$.

$$\begin{aligned}\lim_{k\to 0^+} f_k(t) &= \lim_{k\to 0^+} \frac{u_{4-k}(t) - u_{4+k}(t)}{2k} \\ &= \lim_{k\to 0^+} \frac{H[t-(4-k)] - H[t-(4+k)]}{2k} \\ &\overset{\frac{0}{0}}{\underset{\text{H}}{=}} \lim_{k\to 0^+} \frac{\delta[t-(4-k)] - \delta[t-(4+k)](-1)}{2} \\ &= \delta(t-4).\end{aligned}$$

Now solve the corresponding initial value problem with this as the inhomogeneous term.

$$y'' + y = \delta(t-4), \qquad y(0) = 0, \quad y'(0) = 0$$

Take the Laplace transform of both sides of the ODE.

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{\delta(t-4)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\delta(t-4)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + Y(s) = \int_0^\infty e^{-st}\delta(t-4)\,dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$$[s^2Y(s)] + Y(s) = e^{-s(4)}$$

$$(s^2+1)Y(s) = e^{-4s}$$

Solve for $Y(s)$.

$$Y(s) = \frac{1}{s^2+1}e^{-4s}$$

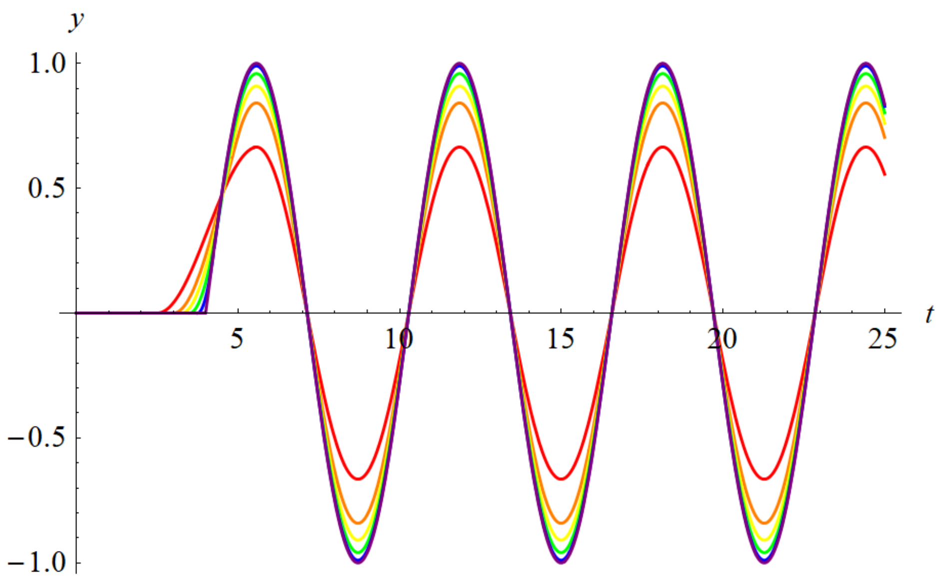
Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned}y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ \phi_0(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}e^{-4s}\right\} \\ &= \sin(t-4)H(t-4)\end{aligned}$$

This result agrees with the one obtained previously in part (b).

Part (d)

The graphs of $\phi(t, 3/2)$, $\phi(t, 1)$, $\phi(t, 3/4)$, $\phi(t, 1/2)$, $\phi(t, 1/4)$, and $\phi_0(t)$ are superimposed on the plot below in red, orange, yellow, green, blue, and purple, respectively.



As the plot shows, the relationship between $\phi(t, k)$ and $\phi_0(t)$ is

$$\lim_{k \to 0} \phi(t, k) = \phi_0(t).$$