

Problem 19

Problems 17 through 22 deal with the effect of a sequence of impulses on an undamped oscillator. Suppose that

$$y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0.$$

For each of the following choices for $f(t)$:

- Try to predict the nature of the solution without solving the problem.
- Test your prediction by finding the solution and drawing its graph.
- Determine what happens after the sequence of impulses ends.

$$f(t) = \sum_{k=1}^{20} \delta(t - k\pi/2)$$

Solution

Because the ODE is linear, the Laplace transform can be used to solve it. The Laplace transform of a function $y(t)$ is defined to be

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^{\infty} e^{-st} y(t) dt.$$

Consequently, the first and second derivatives transform as follows.

$$\begin{aligned} \mathcal{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\ \mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0) \end{aligned}$$

Substitute the function for $f(t)$ and take the Laplace transform of both sides of the ODE.

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\left\{\sum_{k=1}^{20} \delta(t - k\pi/2)\right\}$$

Use the fact that the transform is a linear operator.

$$\begin{aligned} \mathcal{L}\{y''\} + \mathcal{L}\{y\} &= \sum_{k=1}^{20} \mathcal{L}\{\delta(t - k\pi/2)\} \\ [s^2Y(s) - sy(0) - y'(0)] + Y(s) &= \sum_{k=1}^{20} \int_0^{\infty} e^{-st} \delta(t - k\pi/2) dt \end{aligned}$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$$\begin{aligned} [s^2Y(s)] + Y(s) &= \sum_{k=1}^{20} e^{-s(k\pi/2)} \\ (s^2 + 1)Y(s) &= \sum_{k=1}^{20} e^{-k\pi s/2} \end{aligned}$$

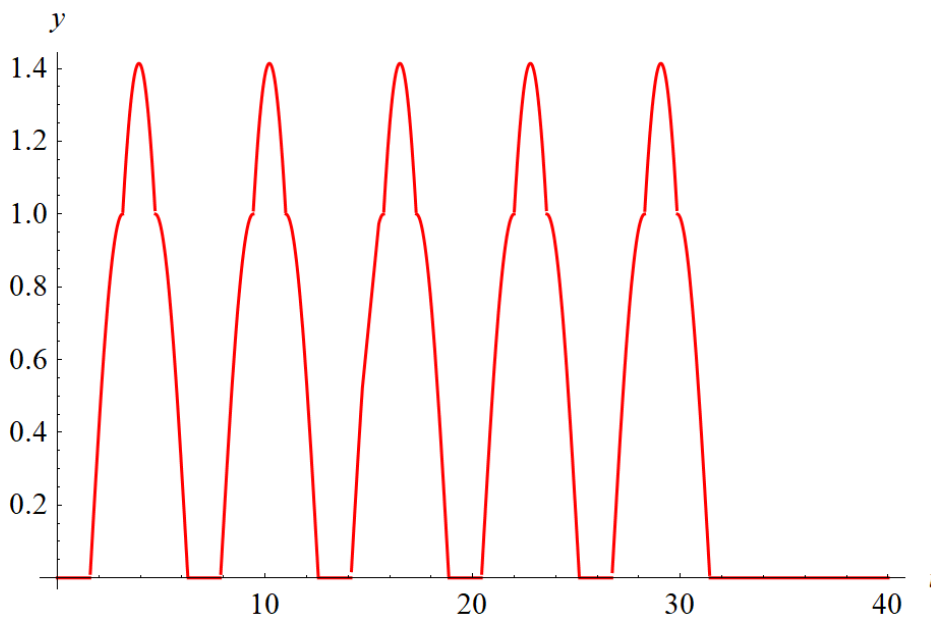
Solve for $Y(s)$.

$$Y(s) = \sum_{k=1}^{20} \frac{1}{s^2 + 1} e^{-k\pi s/2}$$

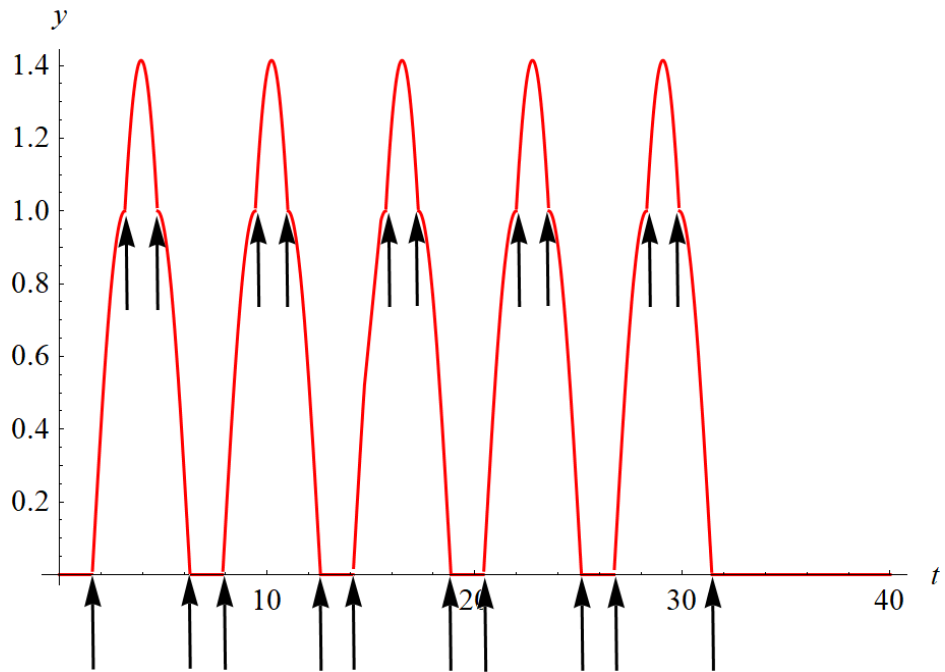
Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\sum_{k=1}^{20} \frac{1}{s^2 + 1} e^{-k\pi s/2}\right\} \\ &= \sum_{k=1}^{20} \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1} e^{-k\pi s/2}\right\} \\ &= \sum_{k=1}^{20} \sin(t - k\pi/2)H(t - k\pi/2) \\ &= \sum_{k=1}^{20} \sin(t - k\pi/2)u_{k\pi/2}(t) \end{aligned}$$

Below is a plot of $y(t)$ versus t up until $t = 40$.



Each of the kinks in the graph represents a point where a delta function strikes. There are ten points on the ground at $y = 0$ and ten points in the air at $y = 1$.



The arrows indicate the time, magnitude, and direction that the delta functions strike.