

Problem 22

Problems 17 through 22 deal with the effect of a sequence of impulses on an undamped oscillator. Suppose that

$$y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0.$$

For each of the following choices for $f(t)$:

- Try to predict the nature of the solution without solving the problem.
- Test your prediction by finding the solution and drawing its graph.
- Determine what happens after the sequence of impulses ends.

$$f(t) = \sum_{k=1}^{40} (-1)^{k+1} \delta(t - 11k/4)$$

Solution

Because the ODE is linear, the Laplace transform can be used to solve it. The Laplace transform of a function $y(t)$ is defined to be

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^{\infty} e^{-st} y(t) dt.$$

Consequently, the first and second derivatives transform as follows.

$$\begin{aligned} \mathcal{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\ \mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0) \end{aligned}$$

Substitute the function for $f(t)$ and take the Laplace transform of both sides of the ODE.

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\left\{\sum_{k=1}^{40} (-1)^{k+1} \delta(t - 11k/4)\right\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \sum_{k=1}^{40} (-1)^{k+1} \mathcal{L}\{\delta(t - 11k/4)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + Y(s) = \sum_{k=1}^{40} (-1)^{k+1} \int_0^{\infty} e^{-st} \delta(t - 11k/4) dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$$[s^2Y(s)] + Y(s) = \sum_{k=1}^{40} (-1)^{k+1} e^{-s(11k/4)}$$

$$(s^2 + 1)Y(s) = \sum_{k=1}^{40} (-1)^{k+1} e^{-11ks/4}$$

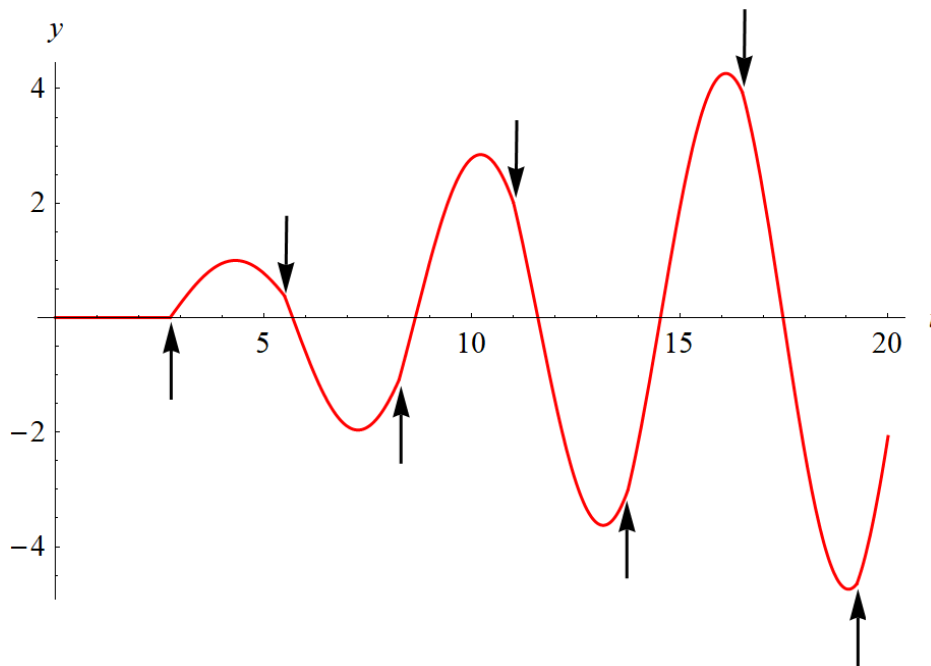
Solve for $Y(s)$.

$$Y(s) = \sum_{k=1}^{40} (-1)^{k+1} \frac{1}{s^2 + 1} e^{-11ks/4}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\sum_{k=1}^{40} (-1)^{k+1} \frac{1}{s^2 + 1} e^{-11ks/4}\right\} \\ &= \sum_{k=1}^{40} (-1)^{k+1} \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1} e^{-11ks/4}\right\} \\ &= \sum_{k=1}^{40} (-1)^{k+1} \sin(t - 11k/4) H(t - 11k/4) \\ &= \sum_{k=1}^{40} (-1)^{k+1} \sin(t - 11k/4) u_{11k/4}(t) \end{aligned}$$

Below is a plot of $y(t)$ versus t up until $t = 20$.



Each of the kinks in the graph represents a point where a delta function strikes. Because of the factor of $(-1)^{k+1}$, the direction that the delta functions act upon alternates. The arrows indicate where and in what direction the delta functions of unit magnitude act.

Below is a plot of $y(t)$ versus t up until $t = 120$.

