# Problem 25

(a) By the method of variation of parameters, show that the solution of the initial value problem

$$y'' + 2y' + 2y = f(t); \qquad y(0) = 0, \quad y'(0) = 0$$

is

$$y = \int_0^t e^{-(t-\tau)} f(\tau) \sin(t-\tau)\, d\tau.$$

(b) Show that if $f(t) = \delta(t-\pi)$, then the solution of part (a) reduces to

$$y = u_\pi(t) e^{-(t-\pi)} \sin(t-\pi).$$

(c) Use a Laplace transform to solve the given initial value problem with $f(t) = \delta(t-\pi)$, and confirm that the solution agrees with the result of part (b).

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**Solution**

## Part (a)

Because the ODE is linear, the general solution can be expressed as a sum of the complementary solution and the particular solution.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + 2y_c' + 2y_c = 0 \tag{1}$$

Since this is a linear homogeneous ODE with constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt}$$

Substitute these expressions into equation (1).

$$r^2 e^{rt} + 2(re^{rt}) + 2(e^{rt}) = 0$$

Divide both sides by $e^{rt}$.

$$r^2 + 2r + 2 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 4(2)}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$r = \{-1 - i, -1 + i\}$$

Two solutions to equation (1) are then $y_c = e^{(-1-i)t}$ and $y_c = e^{(-1+i)t}$. According to the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned}
y_c(t) &= C_1 e^{(-1-i)t} + C_2 e^{(-1+i)t} \\
&= C_1 e^{-t-it} + C_2 e^{-t+it} \\
&= C_1 e^{-t} e^{-it} + C_2 e^{-t} e^{it} \\
&= C_1 e^{-t} (\cos t - i \sin t) + C_2 e^{-t} (\cos t + i \sin t) \\
&= (C_1 + C_2) e^{-t} \cos t + (-iC_1 + iC_2) e^{-t} \sin t \\
&= C_3 e^{-t} \cos t + C_4 e^{-t} \sin t
\end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p'' + 2y_p' + 2y_p = f(t). \tag{2}$$

According to the method of variation of parameters, $y_p$ is found by allowing the parameters in $y_c$ to vary.

$$y_p(t) = C_3(t)e^{-t}\cos t + C_4(t)e^{-t}\sin t$$

Substitute this formula into equation (2).

$$\begin{aligned}[C_3(t)e^{-t}\cos t + C_4(t)e^{-t}\sin t]'' + 2[C_3(t)e^{-t}\cos t + C_4(t)e^{-t}\sin t]' \\ + 2[C_3(t)e^{-t}\cos t + C_4(t)e^{-t}\sin t] = f(t)\end{aligned}$$

Evaluate the derivatives.

$$\begin{aligned}[C_3'(t)e^{-t}\cos t - C_3(t)e^{-t}\cos t - C_3(t)e^{-t}\sin t + C_4'(t)e^{-t}\sin t - C_4(t)e^{-t}\sin t + C_4(t)e^{-t}\cos t]' \\ + 2[C_3'(t)e^{-t}\cos t - C_3(t)e^{-t}\cos t - C_3(t)e^{-t}\sin t + C_4'(t)e^{-t}\sin t - C_4(t)e^{-t}\sin t + C_4(t)e^{-t}\cos t] \\ + 2[C_3(t)e^{-t}\cos t + C_4(t)e^{-t}\sin t] = f(t)\end{aligned}$$

If we set

$$C_3'(t)e^{-t}\cos t + C_4'(t)e^{-t}\sin t = 0, \tag{3}$$

then the previous equation reduces to

$$\begin{aligned}[-C_3(t)e^{-t}\cos t - C_3(t)e^{-t}\sin t - C_4(t)e^{-t}\sin t + C_4(t)e^{-t}\cos t]' \\ + 2[-C_3(t)e^{-t}\cos t - C_3(t)e^{-t}\sin t - C_4(t)e^{-t}\sin t + C_4(t)e^{-t}\cos t] \\ + 2[C_3(t)e^{-t}\cos t + C_4(t)e^{-t}\sin t] = f(t)\end{aligned}$$

$$\begin{aligned}[-C_3'(t)e^{-t}\cos t + C_3(t)e^{-t}\cos t + C_3(t)e^{-t}\sin t - C_3'(t)e^{-t}\sin t + C_3(t)e^{-t}\sin t - C_3(t)e^{-t}\cos t \\ - C_4'(t)e^{-t}\sin t + C_4(t)e^{-t}\sin t - C_4(t)e^{-t}\cos t + C_4'(t)e^{-t}\cos t - C_4(t)e^{-t}\cos t - C_4(t)e^{-t}\sin t] \\ + 2[-C_3(t)e^{-t}\cos t - C_3(t)e^{-t}\sin t - C_4(t)e^{-t}\sin t + C_4(t)e^{-t}\cos t] \\ + 2[C_3(t)e^{-t}\cos t + C_4(t)e^{-t}\sin t] = f(t)\end{aligned}$$

$$-C_3'(t)e^{-t}(\cos t + \sin t) + C_4'(t)e^{-t}(\cos t - \sin t) = f(t). \tag{4}$$

The aim now is to solve equations (3) and (4) for $C_3(t)$ and $C_4(t)$. Solve equation (3) for $C_3'(t)$

$$C_3'(t) = -\frac{\sin t}{\cos t}C_4'(t) \tag{5}$$

and then substitute it into equation (4).

$$-\left[-\frac{\sin t}{\cos t}C_4'(t)\right]e^{-t}(\cos t + \sin t) + C_4'(t)e^{-t}(\cos t - \sin t) = f(t)$$

$$C_4'(t)e^{-t}\left[\frac{\sin t}{\cos t}(\cos t + \sin t) + (\cos t - \sin t)\right] = f(t)$$

$$C_4'(t)e^{-t}\left(\frac{\cos^2 t + \sin^2 t}{\cos t}\right) = f(t)$$

$$C_4'(t)e^{-t}\left(\frac{1}{\cos t}\right) = f(t)$$

Multiply both sides by $e^t \cos t$.

$$C_4'(t) = f(t)e^t \cos t$$

Integrate both sides with respect to $t$, setting the integration constant to zero.

$$C_4(t) = \int^t f(s)e^s \cos s\, ds$$

Now substitute this last formula for $C_4'(t)$ into equation (5) to get $C_3(t)$.

$$C_3'(t) = -f(t)e^t \sin t$$

Integrate both sides with respect to $t$, setting the integration constant to zero.

$$C_3(t) = \int^t -f(s)e^s \sin s\, ds$$

Consequently, the particular solution is

$$\begin{aligned}
y_p(t) &= C_3(t)e^{-t}\cos t + C_4(t)e^{-t}\sin t \\
&= \left[\int^t -f(s)e^s \sin s\, ds\right] e^{-t}\cos t + \left[\int^t f(s)e^s \cos s\, ds\right] e^{-t}\sin t \\
&= -\int^t f(s)e^{s-t}\cos t \sin s\, ds + \int^t f(s)e^{s-t}\sin t \cos s\, ds \\
&= \int^t e^{s-t}[\sin t \cos s - \cos t \sin s] f(s)\, ds \\
&= \int^t e^{-(t-s)}\sin(t-s) f(s)\, ds,
\end{aligned}$$

and the general solution to the original ODE is

$$\begin{aligned}
y(t) &= y_c(t) + y_p(t) \\
&= C_3 e^{-t}\cos t + C_4 e^{-t}\sin t + \int_0^t e^{-(t-s)}\sin(t-s) f(s)\, ds.
\end{aligned}$$

The lower limit of integration is arbitrary and has been set to 0 because that's when the initial conditions are given. Differentiate this solution with respect to $t$, using the Leibnitz rule to differentiate the integral.

$$\begin{aligned}
y'(t) &= -C_3 e^{-t}\cos t - C_3 e^{-t}\sin t - C_4 e^{-t}\sin t + C_4 e^{-t}\cos t + \frac{d}{dt}\int_0^t e^{-(t-s)}\sin(t-s) f(s)\, ds \\
&= -C_3 e^{-t}\cos t - C_3 e^{-t}\sin t - C_4 e^{-t}\sin t + C_4 e^{-t}\cos t \\
&\qquad + \int_0^t \frac{\partial}{\partial t} e^{-(t-s)}\sin(t-s) f(s)\, ds - 0 \cdot e^{-t}\sin(t) f(0) + 1 \cdot e^0 \sin(0) f(t) \\
&= -C_3 e^{-t}\cos t - C_3 e^{-t}\sin t - C_4 e^{-t}\sin t + C_4 e^{-t}\cos t + \int_0^t \frac{\partial}{\partial t}[e^{-(t-s)}\sin(t-s)] f(s)\, ds
\end{aligned}$$

Now apply the initial conditions, $y(0) = 0$ and $y'(0) = 0$ to determine $C_3$ and $C_4$.

$$\begin{aligned} y(0) &= C_3 = 0 \\ y'(0) &= -C_3 + C_4 = 0 \end{aligned}$$

Solving this system yields $C_3 = 0$ and $C_4 = 0$. Therefore,

$$y(t) = \int_0^t e^{-(t-s)} \sin(t-s) f(s)\, ds.$$

## Part (b)

Suppose that $f(t) = \delta(t - \pi)$.

$$y(t) = \int_0^t e^{-(t-s)} \sin(t-s) \delta(s-\pi)\, ds$$

If $\pi$ lies within the interval of $(0, t)$, then this integral evaluates to $e^{-(t-\pi)} \sin(t-\pi)$. If $\pi$ lies outside this interval, then it evaluates to zero. That is,

$$\begin{aligned} y(t) &= \begin{cases} e^{-(t-\pi)} \sin(t-\pi) & \text{if } \pi < t \\ 0 & \text{if } \pi > t \end{cases} \\ &= e^{-(t-\pi)} \sin(t-\pi) H(t-\pi) \\ &= e^{-(t-\pi)} \sin(t-\pi) u_\pi(t). \end{aligned}$$

## Part (c)

$$y'' + 2y' + 2y = \delta(t - \pi)$$

Because the ODE is linear, the Laplace transform can be used to solve it. The Laplace transform of a function $y(t)$ is defined to be

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

As a result, the first and second derivatives transform as follows.

$$\begin{aligned} \mathcal{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\ \mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0) \end{aligned}$$

Take the Laplace transform of both sides of the ODE.

$$\mathcal{L}\{y'' + 2y' + 2y\} = \mathcal{L}\{\delta(t-\pi)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{\delta(t-\pi)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + 2[Y(s)] = \int_0^\infty e^{-st} \delta(t-\pi)\, dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$$[s^2Y(s)] + 2[sY(s)] + 2[Y(s)] = e^{-s(\pi)}$$

$$(s^2 + 2s + 2)Y(s) = e^{-\pi s}$$

Solve for $Y(s)$ and write it in terms of known transforms.

$$\begin{aligned} Y(s) &= \frac{1}{s^2 + 2s + 2} e^{-\pi s} \\ &= \frac{1}{s^2 + 2s + 1 + 2 - 1} e^{-\pi s} \\ &= \frac{1}{(s+1)^2 + 1} e^{-\pi s} \end{aligned}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2 + 1} e^{-\pi s}\right\} \\ &= e^{-(t-\pi)} \sin(t - \pi) H(t - \pi) \\ &= e^{-(t-\pi)} \sin(t - \pi) u_\pi(t) \end{aligned}$$

This result agrees with the one from part (b).