Problem 25

(a) By the method of variation of parameters, show that the solution of the initial value problem
\[ y'' + 2y' + 2y = f(t); \quad y(0) = 0, \quad y'(0) = 0 \]
is
\[ y = \int_0^t e^{-(t-\tau)} f(\tau) \sin(t - \tau) d\tau. \]

(b) Show that if \( f(t) = \delta(t - \pi) \), then the solution of part (a) reduces to
\[ y = u_\pi(t)e^{-(t-\pi)} \sin(t - \pi). \]

(c) Use a Laplace transform to solve the given initial value problem with \( f(t) = \delta(t - \pi) \), and confirm that the solution agrees with the result of part (b).

Solution

Part (a)

Because the ODE is linear, the general solution can be expressed as a sum of the complementary solution and the particular solution.

\[ y(t) = y_c(t) + y_p(t) \]
The complementary solution satisfies the associated homogeneous equation.

\[ y'' + 2y' + 2y = 0 \quad (1) \]

Since this is a linear homogeneous ODE with constant coefficients, the solution is of the form \( y_c = e^{rt} \).

\[ y_c = e^{rt} \rightarrow y'_c = re^{rt} \rightarrow y''_c = r^2e^{rt} \]

Substitute these expressions into equation (1).

\[ r^2e^{rt} + 2(re^{rt}) + 2(e^{rt}) = 0 \]

Divide both sides by \( e^{rt} \).

\[ r^2 + 2r + 2 = 0 \]

\[ r = \frac{-2 \pm \sqrt{4-4(2)}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i \]

Two solutions to equation (1) are then \( y_c = e^{(-1-i)t} \) and \( y_c = e^{(-1+i)t} \). According to the principle of superposition, the general solution is a linear combination of these two.

\[ y_c(t) = C_1e^{(-1-i)t} + C_2e^{(-1+i)t} \]
\[ = C_1e^{-t-it} + C_2e^{-t+it} \]
\[ = C_1e^{-t}e^{-it} + C_2e^{-t}e^{it} \]
\[ = C_1e^{-t}(\cos t - i \sin t) + C_2e^{-t}(\cos t + i \sin t) \]
\[ = (C_1 + C_2)e^{-t} \cos t + (-iC_1 + iC_2)e^{-t} \sin t \]
\[ = C_3e^{-t} \cos t + C_4e^{-t} \sin t \]

www.stemjock.com
On the other hand, the particular solution satisfies
\[ y''_p + 2y'_p + 2y_p = f(t). \]  
(2)

According to the method of variation of parameters, \( y_p \) is found by allowing the parameters in \( y_c \) to vary.
\[ y_p(t) = C_3(t)e^{-t} \cos t + C_4(t)e^{-t} \sin t \]

Substitute this formula into equation (2).
\[
[C_3(t)e^{-t} \cos t + C_4(t)e^{-t} \sin t]' + 2[C_3(t)e^{-t} \cos t + C_4(t)e^{-t} \sin t]' + 2[C_3(t)e^{-t} \cos t + C_4(t)e^{-t} \sin t] = f(t)
\]

Evaluate the derivatives.
\[
[C_3'(t)e^{-t} \cos t - C_3(t)e^{-t} \cos t - C_3(t)e^{-t} \sin t + C_4'(t)e^{-t} \sin t - C_4(t)e^{-t} \sin t + C_4(t)e^{-t} \cos t]' \\
+ 2[C_3'(t)e^{-t} \cos t - C_3(t)e^{-t} \cos t - C_3(t)e^{-t} \sin t + C_4'(t)e^{-t} \sin t - C_4(t)e^{-t} \sin t + C_4(t)e^{-t} \cos t] \\
+ 2[C_3(t)e^{-t} \cos t + C_4(t)e^{-t} \sin t] = f(t)
\]

If we set
\[ C_3'(t)e^{-t} \cos t + C_4'(t)e^{-t} \sin t = 0, \]
then the previous equation reduces to
\[
[-C_3(t)e^{-t} \cos t - C_3(t)e^{-t} \sin t - C_4(t)e^{-t} \sin t + C_4(t)e^{-t} \cos t]' \\
+ 2[-C_3(t)e^{-t} \cos t - C_3(t)e^{-t} \sin t - C_4(t)e^{-t} \sin t + C_4(t)e^{-t} \cos t] \\
+ 2[-C_3(t)e^{-t} \cos t + C_4(t)e^{-t} \sin t] = f(t)
\]

\[
[-C_3'(t)e^{-t} \cos t + C_3(t)e^{-t} \cos t + C_3(t)e^{-t} \sin t - C_3'(t)e^{-t} \sin t - C_3(t)e^{-t} \sin t - C_3(t)e^{-t} \cos t \\
- C_4'(t)e^{-t} \sin t + C_4(t)e^{-t} \sin t - C_4(t)e^{-t} \cos t + C_4'(t)e^{-t} \cos t - C_4(t)e^{-t} \cos t - C_4(t)e^{-t} \sin t] \\
+ 2[-C_3(t)e^{-t} \cos t - C_3(t)e^{-t} \sin t - C_4(t)e^{-t} \sin t + C_4(t)e^{-t} \cos t] \\
+ 2[-C_3(t)e^{-t} \cos t + C_4(t)e^{-t} \sin t] = f(t)
\]

\[ -C_3'(t)e^{-t} \cos t + C_4'(t)e^{-t} \cos t + C_3'(t)e^{-t} \sin t - C_3(t)e^{-t} \sin t - C_3(t)e^{-t} \cos t \\
- C_4'(t)e^{-t} \sin t + C_4(t)e^{-t} \sin t - C_4(t)e^{-t} \cos t + C_4'(t)e^{-t} \cos t - C_4(t)e^{-t} \cos t - C_4(t)e^{-t} \sin t] = f(t)
\]

(4)

The aim now is to solve equations (3) and (4) for \( C_3(t) \) and \( C_4(t) \). Solve equation (3) for \( C_3'(t) \)
\[ C_3'(t) = -\frac{\sin t}{\cos t} C_4'(t) \]  
(5)

and then substitute it into equation (4).
\[ -\left[ -\frac{\sin t}{\cos t} C_4'(t) \right] e^{-t} (\cos t + \sin t) + C_4'(t) e^{-t} (\cos t - \sin t) = f(t) \]
\[ C_4'(t) e^{-t} \left[ \frac{\sin t}{\cos t} (\cos t + \sin t) + (\cos t - \sin t) \right] = f(t) \]
\[ C_4'(t) e^{-t} \left( \frac{\cos^2 t + \sin^2 t}{\cos t} \right) = f(t) \]
\[ C_4'(t)e^{-t}\left(\frac{1}{\cos t}\right) = f(t) \]

Multiply both sides by \(e^t \cos t\).
\[ C_4'(t) = f(t)e^t \cos t \]
Integrate both sides with respect to \(t\), setting the integration constant to zero.
\[ C_4(t) = \int f(s)e^s \cos s \, ds \]

Now substitute this last formula for \(C_4'(t)\) into equation (5) to get \(C_3(t)\).
\[ C_3'(t) = -f(t)e^t \sin t \]
Integrate both sides with respect to \(t\), setting the integration constant to zero.
\[ C_3(t) = \int -f(s)e^s \sin s \, ds \]

Consequently, the particular solution is
\[ y_p(t) = C_3(t)e^{-t} \cos t + C_4(t)e^{-t} \sin t \]
\[ = \int e^{-t} \cos t - f(s)e^s \sin s \, ds \]
\[ = -\int f(s)e^{s-t} \cos t \sin s \, ds + \int f(s)e^{s-t} \sin t \cos s \, ds \]
\[ = \int e^{s-t}[\sin t \cos s - \cos t \sin s]f(s) \, ds \]
\[ = \int e^{-(t-s)} \sin(t-s)f(s) \, ds, \]

and the general solution to the original ODE is
\[ y(t) = y_c(t) + y_p(t) \]
\[ = C_3e^{-t} \cos t + C_4e^{-t} \sin t + \int e^{-(t-s)} \sin(t-s)f(s) \, ds. \]

The lower limit of integration is arbitrary and has been set to 0 because that’s when the initial conditions are given. Differentiate this solution with respect to \(t\), using the Leibnitz rule to differentiate the integral.
\[ y'(t) = -C_3e^{-t} \cos t - C_3e^{-t} \sin t - C_4e^{-t} \sin t + C_4e^{-t} \cos t + \frac{d}{dt} \left[\int e^{-(t-s)} \sin(t-s)f(s) \, ds\right] \]
\[ = -C_3e^{-t} \cos t - C_3e^{-t} \sin t - C_4e^{-t} \sin t + C_4e^{-t} \cos t \]
\[ + \int_0^t \frac{\partial}{\partial t} e^{-(t-s)} \sin(t-s)f(s) \, ds \]
\[ = -C_3e^{-t} \cos t - C_3e^{-t} \sin t - C_4e^{-t} \sin t + C_4e^{-t} \cos t + \int_0^t \frac{\partial}{\partial t} [e^{-(t-s)} \sin(t-s)]f(s) \, ds \]

www.stemjock.com
Now apply the initial conditions, \( y(0) = 0 \) and \( y'(0) = 0 \) to determine \( C_3 \) and \( C_4 \).

\[
\begin{align*}
y(0) &= C_3 = 0 \\
y'(0) &= -C_3 + C_4 = 0
\end{align*}
\]

Solving this system yields \( C_3 = 0 \) and \( C_4 = 0 \). Therefore,

\[
y(t) = \int_0^t e^{-(t-s)} \sin(t-s) f(s) \, ds.
\]

**Part (b)**

Suppose that \( f(t) = \delta(t - \pi) \).

\[
y(t) = \int_0^t e^{-(t-s)} \sin(t-s) \delta(s - \pi) \, ds
\]

If \( \pi \) lies within the interval of \((0,t)\), then this integral evaluates to \( e^{-(t-\pi)} \sin(t-\pi) \). If \( \pi \) lies outside this interval, then it evaluates to zero. That is,

\[
y(t) = \begin{cases} 
  e^{-(t-\pi)} \sin(t-\pi) & \text{if } \pi < t \\
  0 & \text{if } \pi > t
\end{cases}
= e^{-(t-\pi)} \sin(t-\pi) H(t - \pi)
= e^{-(t-\pi)} \sin(t-\pi) u_\pi(t).
\]

**Part (c)**

\[
y'' + 2y' + 2y = \delta(t - \pi)
\]

Because the ODE is linear, the Laplace transform can be used to solve it. The Laplace transform of a function \( y(t) \) is defined to be

\[
Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t) \, dt.
\]

As a result, the first and second derivatives transform as follows.

\[
\begin{align*}
\mathcal{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\
\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0)
\end{align*}
\]

Take the Laplace transform of both sides of the ODE.

\[
\mathcal{L}\{y'' + 2y' + 2y\} = \mathcal{L}\{\delta(t - \pi)\}
\]

Use the fact that the transform is a linear operator.

\[
\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{\delta(t - \pi)\}
\]

\[
[s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + 2[Y(s)] = \int_0^\infty e^{-st} \delta(t - \pi) \, dt
\]
Plug in the initial conditions, \(y(0) = 0\) and \(y'(0) = 0\).

\[
[s^2Y(s)] + 2[sY(s)] + 2[Y(s)] = e^{-s(\pi)}
\]

\[(s^2 + 2s + 2)Y(s) = e^{-\pi s}\]

Solve for \(Y(s)\) and write it in terms of known transforms.

\[
Y(s) = \frac{1}{s^2 + 2s + 2}e^{-\pi s}
\]

\[
= \frac{1}{s^2 + 2s + 1 + 2 - 1}e^{-\pi s}
\]

\[
= \frac{1}{(s + 1)^2 + 1}e^{-\pi s}
\]

Now take the inverse Laplace transform of \(Y(s)\) to get \(y(t)\).

\[
y(t) = L^{-1}\{Y(s)\}
\]

\[
= L^{-1}\left\{\frac{1}{(s + 1)^2 + 1}e^{-\pi s}\right\}
\]

\[
= e^{-(t-\pi)} \sin(t - \pi) H(t - \pi)
\]

\[
= e^{-(t-\pi)} \sin(t - \pi) u_\pi(t)
\]

This result agrees with the one from part (b).