Problem 7

In each of Problems 1 through 12:

(a) Find the solution of the given initial value problem.

(b) Draw a graph of the solution.

\[ y'' + y = \delta(t - 2\pi) \cos t; \quad y(0) = 0, \quad y'(0) = 1 \]

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function \( y(t) \) is defined here as

\[
Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st}y(t)\,dt.
\]

Consequently, the first and second derivatives transform as follows.

\[
\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)
\]

\[
\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)
\]

Apply the Laplace transform to both sides of the ODE.

\[
\mathcal{L}\{y'' + y\} = \mathcal{L}\{\delta(t - 2\pi) \cos t\}
\]

Use the fact that the transform is a linear operator.

\[
\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\delta(t - 2\pi) \cos t\}
\]

\[
[s^2Y(s) - sy(0) - y'(0)] + [Y(s)] = \int_0^\infty e^{-st}\delta(t - 2\pi) \cos t\,dt
\]

Plug in the initial conditions, \( y(0) = 0 \) and \( y'(0) = 1 \).

\[
[s^2Y(s) - 1] + [Y(s)] = e^{-s(2\pi)} \cos(2\pi)
\]

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for \( Y \), the transformed solution.

\[
(s^2 + 1)Y(s) - 1 = e^{-2\pi s}
\]

Solve for \( Y(s) \) and write it in terms of known transforms.

\[
Y(s) = \frac{1}{s^2 + 1} + \frac{1}{s^2 + 1}e^{-2\pi s}
\]

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Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

\[
y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1} + \frac{1}{s^2 + 1}e^{-2\pi s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}e^{-2\pi s}\right\} = \sin t + \sin(t - 2\pi)H(t - 2\pi) = \sin t + (\sin t)H(t - 2\pi) = \left[1 + u_{2\pi}(t)\right]\sin t = \left[1 + u_{2\pi}(t)\right]\sin t
\]

Below is a plot of $y(t)$ versus $t$. 

![Graph of $y(t)$ versus $t$.](www.stemjock.com)