Problem 8

In each of Problems 1 through 12:

(a) Find the solution of the given initial value problem.

(b) Draw a graph of the solution.

\[ y'' + 4y = 2\delta(t - \pi/4); \quad y(0) = 0, \quad y'(0) = 0 \]

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function \( y(t) \) is defined here as

\[ Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st}y(t)\,dt. \]

Consequently, the first and second derivatives transform as follows.

\[ \mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0) \]

\[ \mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0) \]

Apply the Laplace transform to both sides of the ODE.

\[ \mathcal{L}\{y'' + 4y\} = \mathcal{L}\{2\delta(t - \pi/4)\} \]

Use the fact that the transform is a linear operator.

\[ \mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = 2\mathcal{L}\{\delta(t - \pi/4)\} \]

\[ [s^2Y(s) - sy(0) - y'(0)] + 4[Y(s)] = 2 \int_0^\infty e^{-st}[\delta(t - \pi/4)]\,dt \]

Plug in the initial conditions, \( y(0) = 0 \) and \( y'(0) = 0 \).

\[ [s^2Y(s)] + 4[Y(s)] = 2e^{-s(\pi/4)} \]

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for \( Y \), the transformed solution.

\[ (s^2 + 4)Y(s) = 2e^{-\pi s/4} \]

Solve for \( Y(s) \).

\[ Y(s) = \frac{2}{s^2 + 4}e^{-\pi s/4} \]

Now take the inverse Laplace transform of \( Y(s) \) to get \( y(t) \).

\[ y(t) = \mathcal{L}^{-1}\{Y(s)\} \]

\[ = \mathcal{L}^{-1}\left\{\frac{2}{s^2 + 4}e^{-\pi s/4}\right\} \]

\[ = \sin(2(t - \pi/4))H(t - \pi/4) \]

\[ = \sin(2t - \pi/2)H(t - \pi/4) \]

\[ = (\cos 2t)H(t - \pi/4) \]

\[ = -u_{\pi/4}(t) \cos 2t \]

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Below is a plot of $y(t)$ versus $t$. 