Problem 13

In each of Problems 13 through 20, express the solution of the given initial value problem in terms of a convolution integral.

$$y'' + \omega^2 y = g(t); \qquad y(0) = 0, \quad y'(0) = 1$$

Solution

Because this ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\,dt.$$

Consequently, the first and second derivatives transform as follows.

$$\begin{aligned} \mathcal{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\ \mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0) \end{aligned}$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' + \omega^2 y\} = \mathcal{L}\{g(t)\}$$

Use the fact that the transform is a linear operator.

$$\begin{gathered} \mathcal{L}\{y''\} + \omega^2\mathcal{L}\{y\} = \mathcal{L}\{g(t)\} \\ [s^2Y(s) - sy(0) - y'(0)] + \omega^2[Y(s)] = G(s) \end{gathered}$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 1$.

$$[s^2Y(s) - 1] + \omega^2[Y(s)] = G(s)$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$(s^2 + \omega^2)Y(s) - 1 = G(s)$$

Solve for $Y(s)$ and write it in terms of known transforms.

$$\begin{aligned} Y(s) &= \frac{1}{s^2 + \omega^2} + \frac{G(s)}{s^2 + \omega^2} \\ &= \frac{1}{\omega}\frac{\omega}{s^2 + \omega^2} + \frac{1}{\omega}\frac{\omega}{s^2 + \omega^2}G(s) \end{aligned}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{\omega}\frac{\omega}{s^2 + \omega^2} + \frac{1}{\omega}\frac{\omega}{s^2 + \omega^2}G(s)\right\} \\ &= \frac{1}{\omega}\mathcal{L}^{-1}\left\{\frac{\omega}{s^2 + \omega^2}\right\} + \frac{1}{\omega}\mathcal{L}^{-1}\left\{\frac{\omega}{s^2 + \omega^2}G(s)\right\} \\ &= \frac{1}{\omega}\sin\omega t + \frac{1}{\omega}\mathcal{L}^{-1}\left\{\frac{\omega}{s^2 + \omega^2}G(s)\right\} \end{aligned}$$

Since we're taking the inverse Laplace transform of a product of two transforms, we can use the convolution theorem. It says that

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(t-\tau)g(\tau)\,d\tau.$$

Therefore,

$$y(t) = \frac{1}{\omega}\sin\omega t + \frac{1}{\omega}\int_0^t \sin[\omega(t-\tau)]g(\tau)\,d\tau.$$