

Problem 14

In each of Problems 13 through 20, express the solution of the given initial value problem in terms of a convolution integral.

$$y'' + 2y' + 2y = \sin \alpha t; \quad y(0) = 0, \quad y'(0) = 0$$

Solution

Because this ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^{\infty} e^{-st} y(t) dt.$$

Consequently, the first and second derivatives transform as follows.

$$\begin{aligned} \mathcal{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\ \mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0) \end{aligned}$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' + 2y' + 2y\} = \mathcal{L}\{\sin \alpha t\}$$

Use the fact that the transform is a linear operator.

$$\begin{aligned} \mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} &= \mathcal{L}\{\sin \alpha t\} \\ [s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + 2[Y(s)] &= \frac{\alpha}{s^2 + \alpha^2} \end{aligned}$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$$[s^2Y(s)] + 2[sY(s)] + 2[Y(s)] = \frac{\alpha}{s^2 + \alpha^2}$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y , the transformed solution.

$$(s^2 + 2s + 2)Y(s) = \frac{\alpha}{s^2 + \alpha^2}$$

Solve for $Y(s)$ and write it in terms of known transforms.

$$\begin{aligned} Y(s) &= \frac{1}{s^2 + 2s + 2} \frac{\alpha}{s^2 + \alpha^2} \\ &= \frac{1}{s^2 + 2s + 1 + 2 - 1} \frac{\alpha}{s^2 + \alpha^2} \\ &= \frac{1}{(s + 1)^2 + 1} \frac{\alpha}{s^2 + \alpha^2} \end{aligned}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{(s + 1)^2 + 1} \frac{\alpha}{s^2 + \alpha^2}\right\} \end{aligned}$$

Since we're taking the inverse Laplace transform of a product of two transforms, we can use the convolution theorem. It says that

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(t-\tau)g(\tau) d\tau.$$

Therefore,

$$y(t) = \int_0^t e^{-(t-\tau)} \sin(t-\tau) \sin \alpha\tau d\tau.$$