# Problem 18

In each of Problems 13 through 20, express the solution of the given initial value problem in terms of a convolution integral.

$$y'' + 3y' + 2y = \cos \alpha t; \qquad y(0) = 1, \quad y'(0) = 0$$

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### Solution

Because this ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\begin{aligned} \mathcal{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\ \mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0) \end{aligned}$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' + 3y' + 2y\} = \mathcal{L}\{\cos \alpha t\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{\cos \alpha t\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + 3[sY(s) - y(0)] + 2[Y(s)] = \mathcal{L}\{\cos \alpha t\}$$

Plug in the initial conditions, $y(0) = 1$ and $y'(0) = 0$.

$$[s^2Y(s) - s] + 3[sY(s) - 1] + 2[Y(s)] = \mathcal{L}\{\cos \alpha t\}$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for $Y$, the transformed solution.

$$(s^2 + 3s + 2)Y(s) - s - 3 = \mathcal{L}\{\cos \alpha t\}$$

Solve for $Y(s)$ and write it in terms of known transforms.

$$\begin{aligned} Y(s) &= \frac{s+3}{s^2+3s+2} + \frac{1}{s^2+3s+2}\mathcal{L}\{\cos \alpha t\} \\ &= \frac{s+3}{(s+1)(s+2)} + \frac{1}{(s+1)(s+2)}\mathcal{L}\{\cos \alpha t\} \\ &= \left(\frac{2}{s+1} - \frac{1}{s+2}\right) + \left(\frac{1}{s+1} - \frac{1}{s+2}\right)\mathcal{L}\{\cos \alpha t\} \end{aligned}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \mathcal{L}^{-1}\left\{\left(\frac{2}{s+1} - \frac{1}{s+2}\right) + \left(\frac{1}{s+1} - \frac{1}{s+2}\right)\mathcal{L}\{\cos\alpha t\}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{2}{s+1} - \frac{1}{s+2}\right\} + \mathcal{L}^{-1}\left\{\left(\frac{1}{s+1} - \frac{1}{s+2}\right)\mathcal{L}\{\cos\alpha t\}\right\} \\
&= 2e^{-t} - e^{-2t} + \mathcal{L}^{-1}\left\{\left(\frac{1}{s+1} - \frac{1}{s+2}\right)\mathcal{L}\{\cos\alpha t\}\right\}
\end{aligned}$$

Since we're taking the inverse Laplace transform of a product of two transforms, we can use the convolution theorem. It says that

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(t-\tau)g(\tau)\,d\tau.$$

Therefore,

$$y(t) = 2e^{-t} - e^{-2t} + \int_0^t [e^{-(t-\tau)} - e^{-2(t-\tau)}]\cos\alpha\tau\,d\tau.$$