Problem 1

Establish the commutative, distributive, and associative properties of the convolution integral.

(a) \( f * g = g * f \)

(b) \( f * (g_1 + g_2) = f * g_1 + f * g_2 \)

(c) \( f * (g * h) = (f * g) * h \)

Solution

Part (a)

Start by using the definition of the convolution integral \( f * g \).

\[
f * g = \int_0^t f(t - \tau)g(\tau) \, d\tau
\]

Make the substitution \( \xi = t - \tau \). Then \( d\xi = -d\tau \).

\[
f * g = \int_0^t f(\xi)g(t - \xi) (-d\xi)
\]

Use the minus sign to flip the limits of integration.

\[
f * g = \int_0^t f(\xi)g(t - \xi) \, d\xi
\]

\[
= \int_0^t g(t - \xi)f(\xi) \, d\xi
\]

\[
= g * f
\]

Part (b)

Start by using the definition of the convolution integral \( f * (g_1 + g_2) \).

\[
f * g = \int_0^t f(t - \tau)[g_1(\tau) + g_2(\tau)] \, d\tau
\]

\[
= \int_0^t [f(t - \tau)g_1(\tau) + f(t - \tau)g_2(\tau)] \, d\tau
\]

\[
= \int_0^t f(t - \tau)g_1(\tau) \, d\tau + \int_0^t f(t - \tau)g_2(\tau) \, d\tau
\]

\[
= f * g_1 + f * g_2
\]

Part (c)

Start by using the definition of the convolution integral \( f * (g * h) \).

\[
f * (g * h) = \int_0^t f(t - \tau)(g * h)(\tau) \, d\tau
\]

\[
= \int_0^t f(t - \tau) \left[ \int_0^\tau g(\tau - \xi)h(\xi) \, d\xi \right] \, d\tau
\]

\[
= \int_0^t \int_0^\tau f(t - \tau)g(\tau - \xi)h(\xi) \, d\xi \, d\tau
\]

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We want to make the substitution \( u = \tau - \xi \) so that \( g \) will be in terms of a single variable. Doing this right now, though, will make \( h \) in terms of more than one variable because \( d\xi \) comes first. Our aim then is to switch the order of integration to make \( \tau \) come first. The current mode of integration in the \( \tau\xi \)-plane is shown below on the left.

Integrate over this domain as shown on the right to switch the order of integration.

\[
f \ast (g \ast h) = \int_0^t \int_0^{t-\xi} f(t - \tau)g(\tau - \xi)h(\xi) \, d\tau \, d\xi
\]

Now make the substitution \( u = \tau - \xi \). Then \( du = d\tau \).

\[
f \ast (g \ast h) = \int_0^t \int_0^{t-\xi} f(t - \xi - u)g(u)h(\xi) \, du \, d\xi
\]

\[
= \int_0^t \left[ \int_0^{t-\xi} f(t - \xi - u)g(u) \, du \right] h(\xi) \, d\xi
\]

\[
= \int_0^t (f \ast g)(t - \xi)h(\xi) \, d\xi
\]

\[
= (f \ast g) \ast h
\]