Problem 10

In each of Problems 8 through 11, find the inverse Laplace transform of the given function by using the convolution theorem.

\[ F(s) = \frac{1}{(s + 1)^2(s^2 + 4)} \]

Solution

Recognize that \( F(s) \) is a product of the two Laplace transforms,

\[ \frac{1}{(s + 1)^2} = \mathcal{L}\{te^{-t}\} \quad \text{and} \quad \frac{1}{s^2 + 4} = \frac{1}{2} \frac{2}{s^2 + 4} = \mathcal{L}\left\{\frac{1}{2} \sin 2t\right\}. \]

According to the convolution theorem, the inverse Laplace transform of a product is a convolution integral.

\[ \mathcal{L}^{-1}\{G(s)H(s)\} = \int_0^t g(t - \tau)h(\tau)\,d\tau \]

\( g \) and \( h \) are the inverse Laplace transforms of \( G \) and \( H \), respectively. Therefore,

\[
\begin{align*}
    f(t) & = \mathcal{L}^{-1}\{F(s)\} \\
    & = \mathcal{L}^{-1}\left\{\frac{1}{(s + 1)^2(s^2 + 4)}\right\} \\
    & = \int_0^t (t - \tau)e^{-(t-\tau)\frac{1}{2} \sin 2\tau}\,d\tau.
\end{align*}
\]