Problem 12

(a) If \( f(t) = t^m \) and \( g(t) = t^n \), where \( m \) and \( n \) are positive integers, show that

\[
f * g = t^{m+n+1} \int_0^1 u^m (1-u)^n \, du.
\]

(b) Use the convolution theorem to show that

\[
\int_0^1 u^m (1-u)^n \, du = \frac{m!n!}{(m+n+1)!}.
\]

(c) Extend the result of part (b) to the case where \( m \) and \( n \) are positive numbers but not necessarily integers.

Solution

Part (a)

Evaluate the convolution of \( f \) and \( g \).

\[
f * g = \int_0^t f(t-\tau)g(\tau) \, d\tau
\]

\[
= \int_0^t (t-\tau)^m (\tau)^n \, d\tau
\]

\[
= \int_0^t \left[ t \left( 1 - \frac{\tau}{t} \right) \right]^m \left[ \frac{\tau}{t} \right]^n \, d\tau
\]

\[
= \int_0^t t^m \left( 1 - \frac{\tau}{t} \right)^m \left( \frac{\tau}{t} \right)^n \, d\tau
\]

\[
= t^{m+n} \int_0^1 \left( 1 - \frac{t}{t} \right)^m \left( \frac{t}{t} \right)^n \, d\tau
\]

Make the substitution \( u = \tau/t \). Then \( du = d\tau/t \).

\[
f * g = t^{m+n} \int_0^1 (1-u)^m u^n (t \, du)
\]

Therefore,

\[
f * g = t^{m+n+1} \int_0^1 u^m (1-u)^n \, du.
\]
Part (b)

According to the convolution theorem, the inverse Laplace transform of a product $F(s)G(s)$ is a convolution integral.

$$
\int_0^t f(t-\tau)g(\tau) \, d\tau = \mathcal{L}^{-1}\{F(s)G(s)\}
$$

Use this result in equation (1).

$$
\hat{f \ast g} = \int_0^t (t-\tau)^m (\tau)^n \, d\tau = \mathcal{L}^{-1}\{\mathcal{L}\{t^m\}\mathcal{L}\{t^n\}\}
$$

$$
= \mathcal{L}^{-1}\left\{ \frac{m!}{s^{m+1}} \right\} \mathcal{L}^{-1}\left\{ \frac{n!}{s^{n+1}} \right\}
$$

$$
= \frac{m!n!}{s^{m+n+1}} \mathcal{L}^{-1}\left\{ \frac{1}{s^{m+n+1}} \right\}
$$

$$
= \frac{m!n!}{(m+n+1)!} \mathcal{L}^{-1}\left\{ \frac{(m+n+1)!}{s^{m+n+1}} \right\}
$$

$$
= \frac{m!n!}{(m+n+1)!} t^{m+n+1}
$$

Therefore, for positive integers,

$$
\int_0^1 u^m (1-u)^n \, du = \frac{m!n!}{(m+n+1)!} t^{m+n+1}
$$

$$
\int_0^1 u^m (1-u)^n \, du = \frac{m!n!}{(m+n+1)!}.
$$

Part (c)

It was shown in part (c) of Problem 30 in Section 6.1 that $\Gamma(n+1) = n!$. The gamma function is an extension of the factorial function in the case that $n$ is positive but not an integer. Use this result here.

$$
\int_0^1 u^m (1-u)^n \, du = \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}
$$

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