Problem 14

In each of Problems 13 through 20, express the solution of the given initial value problem in terms of a convolution integral.

\[ y'' + 2y' + 2y = \sin \alpha t; \quad y(0) = 0, \quad y'(0) = 0 \]

Solution

Because this ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function \( y(t) \) is defined here as

\[ Y(s) = \mathcal{L}\{y(t)\} = \int_{0}^{\infty} e^{-st}y(t) \, dt. \]

Consequently, the first and second derivatives transform as follows.

\[ \mathcal{L}\left\{ \frac{dy}{dt} \right\} = sY(s) - y(0) \]
\[ \mathcal{L}\left\{ \frac{d^2y}{dt^2} \right\} = s^2Y(s) - sy(0) - y'(0) \]

Apply the Laplace transform to both sides of the ODE.

\[ \mathcal{L}\{y'' + 2y' + 2y\} = \mathcal{L}\{\sin \alpha t\} \]

Use the fact that the transform is a linear operator.

\[ \mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{\sin \alpha t\} \]

\[ [s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + 2[Y(s)] = \frac{\alpha}{s^2 + \alpha^2} \]

Plug in the initial conditions, \( y(0) = 0 \) and \( y'(0) = 0 \).

\[ [s^2Y(s)] + 2[sY(s)] + 2[Y(s)] = \frac{\alpha}{s^2 + \alpha^2} \]

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for \( Y \), the transformed solution.

\[ (s^2 + 2s + 2)Y(s) = \frac{\alpha}{s^2 + \alpha^2} \]

Solve for \( Y(s) \) and write it in terms of known transforms.

\[ Y(s) = \frac{1}{s^2 + 2s + 2} \cdot \frac{\alpha}{s^2 + \alpha^2} \]
\[ = \frac{1}{s^2 + 2s + 1 + 2 - 1} \cdot \frac{\alpha}{s^2 + \alpha^2} \]
\[ = \frac{1}{(s + 1)^2 + 1} \cdot \frac{\alpha}{s^2 + \alpha^2} \]

Now take the inverse Laplace transform of \( Y(s) \) to get \( y(t) \).

\[ y(t) = \mathcal{L}^{-1}\{Y(s)\} \]
\[ = \mathcal{L}^{-1}\left\{ \frac{1}{(s + 1)^2 + 1} \cdot \frac{\alpha}{s^2 + \alpha^2} \right\} \]

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Since we’re taking the inverse Laplace transform of a product of two transforms, we can use the convolution theorem. It says that

\[ \mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(t - \tau)g(\tau) \, d\tau. \]

Therefore,

\[ y(t) = \int_0^t e^{-(t-\tau)} \sin(t - \tau) \sin \alpha \tau \, d\tau. \]