Problem 16

In each of Problems 13 through 20, express the solution of the given initial value problem in terms of a convolution integral.

\[ y'' + y' + \frac{5}{4}y = 1 - u_\pi(t); \quad y(0) = 1, \quad y'(0) = -1 \]

Solution

Because this ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function \( y(t) \) is defined here as

\[ Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st}y(t)\, dt. \]

Consequently, the first and second derivatives transform as follows.

\[ \mathcal{L}\left\{ \frac{dy}{dt} \right\} = sY(s) - y(0) \]
\[ \mathcal{L}\left\{ \frac{d^2y}{dt^2} \right\} = s^2Y(s) - sy(0) - y'(0) \]

Apply the Laplace transform to both sides of the ODE.

\[ \mathcal{L}\left\{ y'' + y' + \frac{5}{4}y \right\} = \mathcal{L}\{1 - u_\pi(t)\} \]

Use the fact that the transform is a linear operator.

\[ \mathcal{L}\{y''\} + \mathcal{L}\{y'\} + \frac{5}{4}\mathcal{L}\{y\} = \mathcal{L}\{1 - u_\pi(t)\} \]

\[ [s^2Y(s) - sy(0) - y'(0)] + [sY(s) - y(0)] + \frac{5}{4}[Y(s)] = \mathcal{L}\{1 - u_\pi(t)\} \]

Plug in the initial conditions, \( y(0) = 1 \) and \( y'(0) = -1 \).

\[ [s^2Y(s) - s + 1] + [sY(s) - 1] + \frac{5}{4}[Y(s)] = \mathcal{L}\{1 - u_\pi(t)\} \]

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for \( Y \), the transformed solution.

\[ \left( s^2 + s + \frac{5}{4} \right) Y(s) - s = \mathcal{L}\{1 - u_\pi(t)\} \]

Solve for \( Y(s) \) and write it in terms of known transforms.

\[ Y(s) = \frac{s}{s^2 + s + \frac{5}{4}} + \frac{1}{s^2 + s + \frac{5}{4}} \mathcal{L}\{1 - u_\pi(t)\} \]

\[ = \frac{s}{s^2 + s + \frac{1}{4} + \frac{5}{4} - rac{1}{4}} + \frac{1}{s^2 + s + \frac{1}{4} + \frac{5}{4} - \frac{1}{4}} \mathcal{L}\{1 - u_\pi(t)\} \]

\[ = \frac{s}{(s + \frac{1}{2})^2 + 1} + \frac{1}{(s + \frac{1}{2})^2 + 1} \mathcal{L}\{1 - u_\pi(t)\} \]

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Make it so that \( s + \frac{1}{2} \) appears in the numerator of the first term.

\[
Y(s) = \frac{s + \frac{1}{2} - \frac{1}{2}}{(s + \frac{1}{2})^2 + 1} + \frac{1}{(s + \frac{1}{2})^2 + 1} \mathcal{L}\{1 - u_\pi(t)\}
\]

\[
= \frac{s + \frac{1}{2} - \frac{1}{2}}{(s + \frac{1}{2})^2 + 1} - \frac{1}{2} \frac{1}{(s + \frac{1}{2})^2 + 1} + \frac{1}{(s + \frac{1}{2})^2 + 1} \mathcal{L}\{1 - u_\pi(t)\}
\]

Now take the inverse Laplace transform of \( Y(s) \) to get \( y(t) \).

\[
y(t) = \mathcal{L}^{-1}\{Y(s)\}
\]

\[
= \mathcal{L}^{-1}\left\{ \frac{s + \frac{1}{2} - \frac{1}{2}}{(s + \frac{1}{2})^2 + 1} - \frac{1}{2} \frac{1}{(s + \frac{1}{2})^2 + 1} + \frac{1}{(s + \frac{1}{2})^2 + 1} \mathcal{L}\{1 - u_\pi(t)\} \right\}
\]

\[
= \mathcal{L}^{-1}\left\{ \frac{s + \frac{1}{2}}{(s + \frac{1}{2})^2 + 1} \right\} - \frac{1}{2} \mathcal{L}^{-1}\left\{ \frac{1}{(s + \frac{1}{2})^2 + 1} \right\} + \mathcal{L}^{-1}\left\{ \frac{1}{(s + \frac{1}{2})^2 + 1} \mathcal{L}\{1 - u_\pi(t)\} \right\}
\]

\[
= e^{-t/2} \cos t - \frac{1}{2} e^{-t/2} \sin t + \mathcal{L}^{-1}\left\{ \frac{1}{(s + \frac{1}{2})^2 + 1} \mathcal{L}\{1 - u_\pi(t)\} \right\}
\]

Since we’re taking the inverse Laplace transform of a product of two transforms, we can use the convolution theorem. It says that

\[
\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(t - \tau)g(\tau) \, d\tau.
\]

Therefore,

\[
y(t) = e^{-t/2} \cos t - \frac{1}{2} e^{-t/2} \sin t + \int_0^t e^{-(t-\tau)/2} \sin(t - \tau)[1 - u_\pi(\tau)] \, d\tau.
\]