Problem 17

In each of Problems 13 through 20, express the solution of the given initial value problem in terms of a convolution integral.

\[ y'' + 4y' + 4y = g(t); \quad y(0) = 2, \quad y'(0) = -3 \]

Solution

Because this ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function \( y(t) \) is defined here as

\[
Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st}y(t)\,dt.
\]

Consequently, the first and second derivatives transform as follows.

\[
\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)
\]

\[
\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)
\]

Apply the Laplace transform to both sides of the ODE.

\[
\mathcal{L}\{y'' + 4y' + 4y\} = \mathcal{L}\{g(t)\}
\]

Use the fact that the transform is a linear operator.

\[
\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{g(t)\}
\]

\[
[s^2Y(s) - sy(0) - y'(0)] + 4[sY(s) - y(0)] + 4[Y(s)] = G(s)
\]

Plug in the initial conditions, \( y(0) = 2 \) and \( y'(0) = -3 \).

\[
[s^2Y(s) - 2s + 3] + 4[sY(s) - 2] + 4[Y(s)] = G(s)
\]

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for \( Y \), the transformed solution.

\[
(s^2 + 4s + 4)Y(s) - 2s - 5 = G(s)
\]

Solve for \( Y(s) \) and write it in terms of known transforms.

\[
Y(s) = \frac{2s + 5}{s^2 + 4s + 4} + \frac{1}{s^2 + 4s + 4}G(s)
\]

\[
= \frac{2s + 5}{(s + 2)^2} + \frac{1}{(s + 2)^2}G(s)
\]

Use partial fraction decomposition.

\[
\frac{2s + 5}{(s + 2)^2} = \frac{A}{s + 2} + \frac{B}{(s + 2)^2}
\]

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Multiply both sides by \((s + 2)^2\).
\[
2s + 5 = A(s + 2) + B
\]
Plug in two random values of \(s\) to get a system of two equations for \(A\) and \(B\).
\[
s = 0 : \quad 5 = 2A + B
s = 1 : \quad 7 = 3A + B
\]
Solving this system yields \(A = 2\) and \(B = 1\).
\[
Y(s) = \left[ \frac{2}{s + 2} + \frac{1}{(s + 2)^2} \right] + \frac{1}{(s + 2)^2} G(s)
\]
Now take the inverse Laplace transform of \(Y(s)\) to get \(y(t)\).
\[
y(t) = \mathcal{L}^{-1}\{Y(s)\}
= \mathcal{L}^{-1}\left\{ \left[ \frac{2}{s + 2} + \frac{1}{(s + 2)^2} \right] + \frac{1}{(s + 2)^2} G(s) \right\}
= \mathcal{L}^{-1}\left\{ \frac{2}{s + 2} + \frac{1}{(s + 2)^2} \right\} + \mathcal{L}^{-1}\left\{ \frac{1}{(s + 2)^2} G(s) \right\}
= 2e^{-2t} + te^{-2t} + \mathcal{L}^{-1}\left\{ \frac{1}{(s + 2)^2} G(s) \right\}
\]
Since we’re taking the inverse Laplace transform of a product of two transforms, we can use the convolution theorem. It says that
\[
\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(t - \tau)g(\tau) \, d\tau.
\]
Therefore,
\[
y(t) = 2e^{-2t} + te^{-2t} + \int_0^t (t - \tau)e^{-2(t-\tau)}g(\tau) \, d\tau.
\]