Problem 21

Consider the equation
\[ \phi(t) + \int_0^t k(t - \xi)\phi(\xi)\,d\xi = f(t), \]
in which \( f \) and \( k \) are known functions, and \( \phi \) is to be determined. Since the unknown function \( \phi \) appears under an integral sign, the given equation is called an integral equation; in particular, it belongs to a class of integral equations known as Volterra integral equations. Take the Laplace transform of the given integral equation and obtain an expression for \( L\{\phi(t)\} \) in terms of the transforms \( L\{f(t)\} \) and \( L\{k(t)\} \) of the given functions \( f \) and \( k \). The inverse transform of \( L\{\phi(t)\} \) is the solution of the original integral equation.

Solution

The Laplace transform of a function \( y(t) \) is defined as
\[ Y(s) = L\{y(t)\} = \int_0^\infty e^{-st}y(t)\,dt. \]

Consequently, the convolution theorem for it is
\[ L\left\{ \int_0^t f(t - \tau)g(\tau)\,d\tau \right\} = F(s)G(s). \]

Take the Laplace transform of both sides of the integral equation.
\[ L\left\{ \phi(t) + \int_0^t k(t - \xi)\phi(\xi)\,d\xi \right\} = L\{f(t)\} \]

Use the fact that the transform is linear.
\[ L\{\phi(t)\} + L\left\{ \int_0^t k(t - \xi)\phi(\xi)\,d\xi \right\} = L\{f(t)\} \]

Use the convolution theorem.
\[ L\{\phi(t)\} + L\{k(t)\} L\{\phi(t)\} = L\{f(t)\} \]

Solve for \( L\{\phi(t)\} \).
\[ L\{\phi(t)\} \left( 1 + L\{k(t)\} \right) = L\{f(t)\} \]
\[ L\{\phi(t)\} = \frac{L\{f(t)\}}{1 + L\{k(t)\}} \]

Therefore,
\[ \Phi(s) = \frac{F(s)}{1 + K(s)}, \]