

Problem 10

In each of Problems 8 through 12, proceed as in Problem 7.

- Transform the given system into a single equation of second order.
- Find x_1 and x_2 that also satisfy the given initial conditions.
- Sketch the graph of the solution in the x_1x_2 -plane for $t \geq 0$.

$$\begin{aligned}x_1' &= x_1 - 2x_2, & x_1(0) &= -1 \\x_2' &= 3x_1 - 4x_2, & x_2(0) &= 2\end{aligned}$$

Solution

Solve this first equation for x_2 .

$$x_2 = -\frac{1}{2}x_1' + \frac{1}{2}x_1 \quad \rightarrow \quad x_2' = -\frac{1}{2}x_1'' + \frac{1}{2}x_1'$$

Substitute these formulas into the second equation.

$$\begin{aligned}x_2' &= 3x_1 - 4x_2 \\-\frac{1}{2}x_1'' + \frac{1}{2}x_1' &= 3x_1 - 4\left(-\frac{1}{2}x_1' + \frac{1}{2}x_1\right) \\-\frac{1}{2}x_1'' + \frac{1}{2}x_1' &= 3x_1 + 2x_1' - 2x_1\end{aligned}$$

Bring all terms to the left side and then multiply both sides by -2 .

$$x_1'' + 3x_1' + 2x_1 = 0$$

This is a constant-coefficient linear ODE, so it has solutions of the form $x_1 = e^{rt}$. Determine formulas for the derivatives

$$x_1 = e^{rt} \quad \rightarrow \quad x_1' = re^{rt} \quad \rightarrow \quad x_1'' = r^2e^{rt}$$

and then plug them into the ODE.

$$r^2e^{rt} + 3re^{rt} + 2e^{rt} = 0$$

Divide both sides by e^{rt} .

$$r^2 + 3r + 2 = 0$$

Solve for r .

$$(r + 2)(r + 1) = 0$$

$$r = \{-2, -1\}$$

Two solutions to the ODE for x_1 are $x_1 = e^{-2t}$ and $x_1 = e^{-t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$x_1(t) = C_1e^{-2t} + C_2e^{-t}$$

Take the first derivative.

$$x_1'(t) = -2C_1e^{-2t} - C_2e^{-t}$$

Now calculate x_2 using the formula in the beginning.

$$\begin{aligned} x_2 &= -\frac{1}{2}x_1' + \frac{1}{2}x_1 \\ &= -\frac{1}{2}(-2C_1e^{-2t} - C_2e^{-t}) + \frac{1}{2}(C_1e^{-2t} + C_2e^{-t}) \\ &= \frac{3}{2}C_1e^{-2t} + C_2e^{-t} \end{aligned}$$

Apply the provided initial conditions to determine C_1 and C_2 .

$$x_1(0) = C_1 + C_2 = -1$$

$$x_2(0) = \frac{3}{2}C_1 + C_2 = 2$$

Solving this system of equations yields

$$C_1 = 6 \quad \text{and} \quad C_2 = -7.$$

Therefore,

$$x_1(t) = 6e^{-2t} - 7e^{-t}$$

$$x_2(t) = 9e^{-2t} - 7e^{-t}.$$

Below is a parametric plot of $\{x_1(t), x_2(t)\}$ as t goes from 0 to 50.

