Problem 11

In each of Problems 8 through 12, proceed as in Problem 7.

(a) Transform the given system into a single equation of second order.

(b) Find x_1 and x_2 that also satisfy the given initial conditions.

(c) Sketch the graph of the solution in the x_1x_2-plane for $t \geq 0$.

$$\begin{aligned} x_1' &= 2x_2, \qquad x_1(0) = 3 \\ x_2' &= -2x_1, \qquad x_2(0) = 4 \end{aligned}$$

Solution

Solve this first equation for x_2.

$$x_2 = \frac{1}{2}x_1' \quad \rightarrow \quad x_2' = \frac{1}{2}x_1''$$

Substitute these formulas into the second equation.

$$\begin{aligned} x_2' &= -2x_1 \\ \frac{1}{2}x_1'' &= -2x_1 \end{aligned}$$

Bring all terms to the left side and then multiply both sides by 2.

$$x_1'' + 4x_1 = 0$$

This is a constant-coefficient linear ODE, so it has solutions of the form $x_1 = e^{rt}$. Determine formulas for the derivatives

$$x_1 = e^{rt} \quad \rightarrow \quad x_1' = re^{rt} \quad \rightarrow \quad x_1'' = r^2e^{rt}$$

and then plug them into the ODE.

$$r^2e^{rt} + 4e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^2 + 4 = 0$$

Solve for r.

$$(r + 2i)(r - 2i) = 0$$

$$r = \{-2i, 2i\}$$

Two solutions to the ODE for x_1 are $x_1 = e^{-2it}$ and $x_1 = e^{2it}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned} x_1(t) &= C_1e^{-2it} + C_2e^{2it} \\ &= C_1(\cos 2t - i\sin 2t) + C_2(\cos 2t + i\sin 2t) \\ &= (C_1 + C_2)\cos 2t + (-iC_1 + iC_2)\sin 2t \\ &= C_3\cos 2t + C_4\sin 2t \end{aligned}$$

Take the first derivative.

$$x_1'(t) = -2C_3 \sin 2t + 2C_4 \cos 2t$$

Now calculate x_2 using the formula in the beginning.

$$\begin{aligned} x_2 &= \frac{1}{2}x_1' \\ &= \frac{1}{2}(-2C_3 \sin 2t + 2C_4 \cos 2t) \\ &= -C_3 \sin 2t + C_4 \cos 2t \end{aligned}$$

Apply the provided initial conditions to determine C_1 and C_2.

$$\begin{aligned} x_1(0) &= C_3 = 3 \\ x_2(0) &= C_4 = 4 \end{aligned}$$

Therefore,

$$x_1(t) = 3\cos 2t + 4\sin 2t$$

$$x_2(t) = -3\sin 2t + 4\cos 2t.$$

Below is a parametric plot of $\{x_1(t), x_2(t)\}$ as t goes from 0 to π.