Problem 11

In each of Problems 8 through 12, proceed as in Problem 7.

(a) Transform the given system into a single equation of second order.

(b) Find $x_1$ and $x_2$ that also satisfy the given initial conditions.

(c) Sketch the graph of the solution in the $x_1x_2$-plane for $t \geq 0$.

\[
x'_1 = 2x_2, \quad x_1(0) = 3 \\
x'_2 = -2x_1, \quad x_2(0) = 4
\]

Solution

Solve this first equation for $x_2$.

\[
x_2 = \frac{1}{2} x'_1 \rightarrow x'_2 = \frac{1}{2} x''_1
\]

Substitute these formulas into the second equation.

\[
x'_2 = -2x_1 \\
\frac{1}{2} x''_1 = -2x_1
\]

Bring all terms to the left side and then multiply both sides by 2.

\[
x''_1 + 4x_1 = 0
\]

This is a constant-coefficient linear ODE, so it has solutions of the form $x_1 = e^{rt}$. Determine formulas for the derivatives

\[
x_1 = e^{rt} \rightarrow x'_1 = re^{rt} \rightarrow x''_1 = r^2 e^{rt}
\]

and then plug them into the ODE.

\[
r^2 e^{rt} + 4e^{rt} = 0
\]

Divide both sides by $e^{rt}$.

\[
r^2 + 4 = 0
\]

Solve for $r$.

\[
(r + 2i)(r - 2i) = 0 \\
r = \{-2i, 2i\}
\]

Two solutions to the ODE for $x_1$ are $x_1 = e^{-2it}$ and $x_1 = e^{2it}$. By the principle of superposition, the general solution is a linear combination of these two.

\[
x_1(t) = C_1 e^{-2it} + C_2 e^{2it} \\
= C_1 (\cos 2t - i \sin 2t) + C_2 (\cos 2t + i \sin 2t) \\
= (C_1 + C_2) \cos 2t + (-iC_1 + iC_2) \sin 2t \\
= C_3 \cos 2t + C_4 \sin 2t
\]
Take the first derivative.

\[ x'_1(t) = -2C_3 \sin 2t + 2C_4 \cos 2t \]

Now calculate \( x_2 \) using the formula in the beginning.

\[
\begin{align*}
 x_2 &= \frac{1}{2} x'_1 \\
 &= \frac{1}{2} (-2C_3 \sin 2t + 2C_4 \cos 2t) \\
 &= -C_3 \sin 2t + C_4 \cos 2t
\end{align*}
\]

Apply the provided initial conditions to determine \( C_1 \) and \( C_2 \).

\[
\begin{align*}
 x_1(0) &= C_3 = 3 \\
 x_2(0) &= C_4 = 4
\end{align*}
\]

Therefore,

\[
\begin{align*}
 x_1(t) &= 3 \cos 2t + 4 \sin 2t \\
 x_2(t) &= -3 \sin 2t + 4 \cos 2t.
\end{align*}
\]

Below is a parametric plot of \( \{x_1(t), x_2(t)\} \) as \( t \) goes from 0 to \( \pi \).