Problem 12

In each of Problems 8 through 12, proceed as in Problem 7.

(a) Transform the given system into a single equation of second order.

(b) Find x_1 and x_2 that also satisfy the given initial conditions.

(c) Sketch the graph of the solution in the x_1x_2-plane for $t \geq 0$.

$$\begin{aligned} x_1' &= -0.5x_1 + 2x_2, \qquad & x_1(0) &= -2 \\ x_2' &= -2x_1 - 0.5x_2, \qquad & x_2(0) &= 2 \end{aligned}$$

Solution

Solve this first equation for x_2.

$$x_2 = \frac{1}{2}x_1' + \frac{1}{4}x_1 \quad \rightarrow \quad x_2' = \frac{1}{2}x_1'' + \frac{1}{4}x_1'$$

Substitute these formulas into the second equation.

$$\begin{aligned} x_2' &= -2x_1 - 0.5x_2 \\ \frac{1}{2}x_1'' + \frac{1}{4}x_1' &= -2x_1 - 0.5\left(\frac{1}{2}x_1' + \frac{1}{4}x_1\right) \\ \frac{1}{2}x_1'' + \frac{1}{4}x_1' &= -2x_1 - \frac{1}{4}x_1' - \frac{1}{8}x_1 \end{aligned}$$

Bring all terms to the left side and then multiply both sides by 2.

$$x_1'' + x_1' + \frac{17}{4}x_1 = 0$$

This is a constant-coefficient linear ODE, so it has solutions of the form $x_1 = e^{rt}$. Determine formulas for the derivatives

$$x_1 = e^{rt} \quad \rightarrow \quad x_1' = re^{rt} \quad \rightarrow \quad x_1'' = r^2e^{rt}$$

and then plug them into the ODE.

$$r^2e^{rt} + re^{rt} + \frac{17}{4}e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^2 + r + \frac{17}{4} = 0$$

Solve for r.

$$r = \frac{-1 \pm \sqrt{1 - 17}}{2} = \frac{-1 \pm 4i}{2}$$

$$r = \left\{\frac{-1 - 4i}{2}, \frac{-1 + 4i}{2}\right\}$$

Two solutions to the ODE for x_1 are $x_1 = e^{[(-1-4i)/2]t}$ and $x_1 = e^{[(-1+4i)/2]t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned} x_1(t) &= C_1 e^{[(-1-4i)/2]t} + C_2 e^{[(-1+4i)/2]t} \\ &= C_1 e^{-t/2} e^{-2it} + C_2 e^{-t/2} e^{2it} \\ &= e^{-t/2}[C_1(\cos 2t - i \sin 2t) + C_2(\cos 2t + i \sin 2t)] \\ &= e^{-t/2}[(C_1 + C_2)\cos 2t + (-iC_1 + iC_2)\sin 2t] \\ &= e^{-t/2}(C_3 \cos 2t + C_4 \sin 2t) \end{aligned}$$

Take the first derivative.

$$x_1'(t) = -\frac{1}{2}e^{-t/2}(C_3 \cos 2t + C_4 \sin 2t) + e^{-t/2}(-2C_3 \sin 2t + 2C_4 \cos 2t)$$

Now calculate x_2 using the formula in the beginning.

$$\begin{aligned} x_2 &= \frac{1}{2}x_1' + \frac{1}{4}x_1 \\ &= \frac{1}{2}\left[-\frac{1}{2}e^{-t/2}(C_3 \cos 2t + C_4 \sin 2t) + e^{-t/2}(-2C_3 \sin 2t + 2C_4 \cos 2t)\right] + \frac{1}{4}\left[e^{-t/2}(C_3 \cos 2t + C_4 \sin 2t)\right] \\ &= e^{-t/2}(-C_3 \sin 2t + C_4 \cos 2t) \end{aligned}$$

Apply the provided initial conditions to determine C_1 and C_2.

$$\begin{aligned} x_1(0) &= C_3 = -2 \\ x_2(0) &= C_4 = 2 \end{aligned}$$

Therefore,

$$x_1(t) = e^{-t/2}(-2\cos 2t + 2\sin 2t)$$

$$x_2(t) = e^{-t/2}(2\sin 2t + 2\cos 2t).$$

Below is a parametric plot of $\{x_1(t), x_2(t)\}$ as t goes from 0 to 10.