Problem 12

In each of Problems 8 through 12, proceed as in Problem 7.

(a) Transform the given system into a single equation of second order.
(b) Find $x_1$ and $x_2$ that also satisfy the given initial conditions.
(c) Sketch the graph of the solution in the $x_1x_2$-plane for $t \geq 0$.

\[
\begin{align*}
x_1' &= -0.5x_1 + 2x_2, \quad x_1(0) = -2 \\
x_2' &= -2x_1 - 0.5x_2, \quad x_2(0) = 2
\end{align*}
\]

Solution

Solve this first equation for $x_2$.

\[
x_2 = \frac{1}{2} x_1' + \frac{1}{4} x_1 \quad \Rightarrow \quad x_2' = \frac{1}{2} x_1'' + \frac{1}{4} x_1'
\]

Substitute these formulas into the second equation.

\[
\frac{1}{2} x_1'' + \frac{1}{4} x_1' = -2x_1 - 0.5 \left( \frac{1}{2} x_1' + \frac{1}{4} x_1 \right)
\]

Bring all terms to the left side and then multiply both sides by 2.

\[
x_1'' + x_1' + \frac{17}{4} x_1 = 0
\]

This is a constant-coefficient linear ODE, so it has solutions of the form $x_1 = e^{rt}$. Determine formulas for the derivatives

\[
x_1 = e^{rt} \quad \Rightarrow \quad x_1' = re^{rt} \quad \Rightarrow \quad x_1'' = r^2 e^{rt}
\]

and then plug them into the ODE.

\[
r^2 e^{rt} + re^{rt} + \frac{17}{4} e^{rt} = 0
\]

Divide both sides by $e^{rt}$.

\[
r^2 + r + \frac{17}{4} = 0
\]

Solve for $r$.

\[
r = \frac{-1 \pm \sqrt{1 - 17}}{2} = \frac{-1 \pm 4i}{2}
\]

\[
r = \left\{ \frac{-1 - 4i}{2}, \frac{-1 + 4i}{2} \right\}
\]

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Two solutions to the ODE for $x_1$ are $x_1 = e^{[(−1−4i)/2]t}$ and $x_1 = e^{[(−1+4i)/2]t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$x_1(t) = C_1 e^{[(−1−4i)/2]t} + C_2 e^{[(−1+4i)/2]t}$$

$$= C_1 e^{-t/2} e^{-2it} + C_2 e^{-t/2} e^{2it}$$

$$= e^{-t/2} [C_1 (cos 2t - i sin 2t) + C_2 (cos 2t + i sin 2t)]$$

$$= e^{-t/2} [(C_1 + C_2) cos 2t + (-iC_1 + iC_2) sin 2t]$$

$$= e^{-t/2} (C_3 cos 2t + C_4 sin 2t)$$

Take the first derivative.

$$x_1'(t) = -\frac{1}{2} e^{-t/2} (C_3 cos 2t + C_4 sin 2t) + e^{-t/2} (-2C_3 sin 2t + 2C_4 cos 2t)$$

Now calculate $x_2$ using the formula in the beginning.

$$x_2 = \frac{1}{2} x_1' + \frac{1}{4} x_1$$

$$= \frac{1}{2} \left[ -\frac{1}{2} e^{-t/2} (C_3 cos 2t + C_4 sin 2t) + e^{-t/2} (-2C_3 sin 2t + 2C_4 cos 2t) \right] + \frac{1}{4} \left[ e^{-t/2} (C_3 cos 2t + C_4 sin 2t) \right]$$

$$= e^{-t/2} (-C_3 sin 2t + C_4 cos 2t)$$

Apply the provided initial conditions to determine $C_1$ and $C_2$.

$$x_1(0) = C_3 = -2$$

$$x_2(0) = C_4 = 2$$

Therefore,

$$x_1(t) = e^{-t/2} (-2 cos 2t + 2 sin 2t)$$

$$x_2(t) = e^{-t/2} (2 sin 2t + 2 cos 2t).$$
Below is a parametric plot of \( \{x_1(t), x_2(t)\} \) as \( t \) goes from 0 to 10.