Problem 14

Show that if \( a_{11}, a_{12}, a_{21}, \) and \( a_{22} \) are constants with \( a_{12} \) and \( a_{21} \) not both zero, and if the functions \( g_1 \) and \( g_2 \) are differentiable, then the initial value problem

\[
\begin{align*}
x_1' &= a_{11}x_1 + a_{12}x_2 + g_1(t), \quad x_1(0) = x_1^0 \\
x_2' &= a_{21}x_1 + a_{22}x_2 + g_2(t), \quad x_2(0) = x_2^0
\end{align*}
\]

can be transformed into an initial value problem for a single second order equation. Can the same procedure be carried out if \( a_{11}, \ldots, a_{22} \) are functions of \( t \)?

Solution

Solve this first equation for \( x_2 \)

\[
x_2(t) = \frac{1}{a_{12}} x'_1 - \frac{a_{11}}{a_{12}} x_1 - \frac{1}{a_{12}} g_1(t)
\]

and then substitute it into the second equation.

\[
\frac{d}{dt} \left[ \frac{1}{a_{12}} x'_1 - \frac{a_{11}}{a_{12}} x_1 - \frac{1}{a_{12}} g_1(t) \right] = a_{21} x_1 + a_{22} \left[ \frac{1}{a_{12}} x'_1 - \frac{a_{11}}{a_{12}} x_1 - \frac{1}{a_{12}} g_1(t) \right] + g_2(t)
\]

Simplify the right side.

\[
\frac{d}{dt} \left[ \frac{1}{a_{12}} x'_1 - \frac{a_{11}}{a_{12}} x_1 - \frac{1}{a_{12}} g_1(t) \right] = \frac{a_{22}}{a_{12}} x'_1 + \frac{a_{12} a_{21} - a_{11} a_{22}}{a_{12}} x_1 - \frac{a_{22}}{a_{12}} g_1(t) + g_2(t) \quad (1)
\]

Assuming that \( a_{11}, a_{12}, a_{21}, \) and \( a_{22} \) are constants, equation (1) reduces to

\[
\frac{1}{a_{12}} x''_1 - \frac{a_{11}}{a_{12}} x'_1 - \frac{1}{a_{12}} g'_1(t) = \frac{a_{22}}{a_{12}} x'_1 + \frac{a_{12} a_{21} - a_{11} a_{22}}{a_{12}} x_1 - \frac{a_{22}}{a_{12}} g_1(t) + g_2(t).
\]

Bring all terms with \( g_1 \) and \( g_2 \) to the right side and all the rest to the left side.

\[
\frac{1}{a_{12}} x''_1 - \frac{a_{11} + a_{22}}{a_{12}} x'_1 - \frac{a_{12} a_{21} - a_{11} a_{22}}{a_{12}} x_1 = \frac{1}{a_{12}} g'_1(t) - \frac{a_{22}}{a_{12}} g_1(t) + g_2(t).
\]

Finally, multiply both sides by \( a_{12} \).

\[
\boxed{x''_1 - (a_{11} + a_{22}) x'_1 - (a_{12} a_{21} - a_{11} a_{22}) x_1 = g'_1(t) - a_{22} g_1(t) + a_{12} g_2(t)}.
\]

The initial conditions associated with this second-order ODE for \( x_1 \) are

\[
\boxed{x_1(0) = x_1^0 \quad \text{and} \quad x_2(0) = \frac{1}{a_{12}} x'_1(0) - \frac{a_{11}}{a_{12}} x_1(0) - \frac{1}{a_{12}} g_1(0) = x_2^0 \Rightarrow x'_1(0) = a_{12} x_2^0 + a_{11} x_1^0 + g_1(0).}
\]
Repeat equation (1) here.

\[
\frac{d}{dt} \left[ \frac{1}{a_{12}} x_1' - \frac{a_{11}}{a_{12}} x_1 - \frac{1}{a_{12}} g_1(t) \right] = \frac{a_{22}}{a_{12}} x_1' + \frac{a_{12}a_{21} - a_{11}a_{22}}{a_{12}} x_1 - \frac{a_{22}}{a_{12}} g_1(t) + g_2(t) \tag{1}
\]

Assuming that \(a_{11}, a_{12}, a_{21},\) and \(a_{22}\) are not constants, equation (1) reduces to

\[
-\frac{a_{12}'}{a_{12}^2} x_1' + \frac{1}{a_{12}} x_1'' - \frac{a_{11}'}{a_{12} a_{12}'} - \frac{a_{11} a_{12}'}{a_{12}^2} x_1 - \frac{a_{11}}{a_{12}} x_1' + \frac{a_{12}'}{a_{12}} g_1(t) - \frac{1}{a_{12}} g_1'(t) = \frac{a_{22}}{a_{12}} x_1' + \frac{a_{12}a_{21} - a_{11}a_{22}}{a_{12}} x_1 - \frac{a_{22}}{a_{12}} g_1(t) + g_2(t).
\]

Multiply both sides by \(a_{12}\).

\[
-\frac{a_{12}'}{a_{12}} x_1' + x_1'' - \frac{a_{11}'}{a_{12}} x_1 - \frac{a_{11} a_{12}'}{a_{12}} x_1 - \frac{a_{11}}{a_{12}} x_1' - \frac{a_{12}'}{a_{12}} g_1(t) - \frac{1}{a_{12}} g_1'(t) = \frac{a_{22}}{a_{12}} x_1' + \frac{a_{12}a_{21} - a_{11}a_{22}}{a_{12}} x_1 - \frac{a_{22}}{a_{12}} g_1(t) + a_{12}g_2(t)
\]

Bring all terms with \(g_1\) and \(g_2\) to the right side and all the rest to the left side.

\[
x_1'' - \left( \frac{a_{11}'}{a_{12}} + \frac{a_{11} a_{12}'}{a_{12}^2} + \frac{a_{12} a_{21} - a_{11} a_{22}}{a_{12}} \right) x_1' = g_1'(t) - \left( \frac{a_{12}'}{a_{12}} + a_{22} \right) g_1(t) + a_{12}g_2(t)
\]

This is the ODE that \(x_1\) satisfies if \(a_{11}, a_{12}, a_{21},\) and \(a_{22}\) are time-dependent (the same procedure can be carried out). Notice that if they are constants, the primed \(a\)-terms vanish, and the ODE reduces to the previous boxed result. The initial conditions that \(x_1\) must satisfy are the same as before.