Problem 14

Show that if a_{11}, a_{12}, a_{21}, and a_{22} are constants with a_{12} and a_{21} not both zero, and if the functions g_1 and g_2 are differentiable, then the initial value problem

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + g_1(t), \qquad x_1(0) = x_1^0 \\ x_2' &= a_{21}x_1 + a_{22}x_2 + g_2(t), \qquad x_2(0) = x_2^0 \end{aligned}$$

can be transformed into an initial value problem for a single second order equation. Can the same procedure be carried out if $a_{11}, \ldots, a_{22}$ are functions of t?

Solution

Solve this first equation for x_2

$$x_2(t) = \frac{1}{a_{12}}x_1' - \frac{a_{11}}{a_{12}}x_1 - \frac{1}{a_{12}}g_1(t)$$

and then substitute it into the second equation.

$$\frac{d}{dt}\left[\frac{1}{a_{12}}x_1' - \frac{a_{11}}{a_{12}}x_1 - \frac{1}{a_{12}}g_1(t)\right] = a_{21}x_1 + a_{22}\left[\frac{1}{a_{12}}x_1' - \frac{a_{11}}{a_{12}}x_1 - \frac{1}{a_{12}}g_1(t)\right] + g_2(t)$$

Simplify the right side.

$$\frac{d}{dt}\left[\frac{1}{a_{12}}x_1' - \frac{a_{11}}{a_{12}}x_1 - \frac{1}{a_{12}}g_1(t)\right] = \frac{a_{22}}{a_{12}}x_1' + \frac{a_{12}a_{21} - a_{11}a_{22}}{a_{12}}x_1 - \frac{a_{22}}{a_{12}}g_1(t) + g_2(t) \tag{1}$$

Assuming that a_{11}, a_{12}, a_{21}, and a_{22} are constants, equation (1) reduces to

$$\frac{1}{a_{12}}x_1'' - \frac{a_{11}}{a_{12}}x_1' - \frac{1}{a_{12}}g_1'(t) = \frac{a_{22}}{a_{12}}x_1' + \frac{a_{12}a_{21} - a_{11}a_{22}}{a_{12}}x_1 - \frac{a_{22}}{a_{12}}g_1(t) + g_2(t).$$

Bring all terms with g_1 and g_2 to the right side and all the rest to the left side.

$$\frac{1}{a_{12}}x_1'' - \frac{a_{11} + a_{22}}{a_{12}}x_1' - \frac{a_{12}a_{21} - a_{11}a_{22}}{a_{12}}x_1 = \frac{1}{a_{12}}g_1'(t) - \frac{a_{22}}{a_{12}}g_1(t) + g_2(t).$$

Finally, multiply both sides by a_{12}.

$$\boxed{x_1'' - (a_{11} + a_{22})x_1' - (a_{12}a_{21} - a_{11}a_{22})x_1 = g_1'(t) - a_{22}g_1(t) + a_{12}g_2(t).}$$

The initial conditions associated with this second-order ODE for x_1 are

$$\boxed{x_1(0) = x_1^0} \quad \text{and} \quad x_2(0) = \frac{1}{a_{12}}x_1'(0) - \frac{a_{11}}{a_{12}}x_1(0) - \frac{1}{a_{12}}g_1(0) = x_2^0 \quad \Rightarrow \quad \boxed{x_1'(0) = a_{12}x_2^0 + a_{11}x_1^0 + g_1(0).}$$

Repeat equation (1) here.

$$\frac{d}{dt}\left[\frac{1}{a_{12}}x_1' - \frac{a_{11}}{a_{12}}x_1 - \frac{1}{a_{12}}g_1(t)\right] = \frac{a_{22}}{a_{12}}x_1' + \frac{a_{12}a_{21} - a_{11}a_{22}}{a_{12}}x_1 - \frac{a_{22}}{a_{12}}g_1(t) + g_2(t) \qquad (1)$$

Assuming that a_{11}, a_{12}, a_{21}, and a_{22} are not constants, equation (1) reduces to

$$\begin{aligned} -\frac{a_{12}'}{a_{12}^2}x_1' + \frac{1}{a_{12}}x_1'' - \frac{a_{11}'a_{12} - a_{11}a_{12}'}{a_{12}^2}x_1 - \frac{a_{11}}{a_{12}}x_1' + \frac{a_{12}'}{a_{12}^2}g_1(t) - \frac{1}{a_{12}}g_1'(t) \\ = \frac{a_{22}}{a_{12}}x_1' + \frac{a_{12}a_{21} - a_{11}a_{22}}{a_{12}}x_1 - \frac{a_{22}}{a_{12}}g_1(t) + g_2(t). \end{aligned}$$

Multiply both sides by a_{12}.

$$\begin{aligned} -\frac{a_{12}'}{a_{12}}x_1' + x_1'' - \frac{a_{11}'a_{12} - a_{11}a_{12}'}{a_{12}}x_1 - a_{11}x_1' + \frac{a_{12}'}{a_{12}}g_1(t) - g_1'(t) \\ = a_{22}x_1' + \frac{a_{12}(a_{12}a_{21} - a_{11}a_{22})}{a_{12}}x_1 - a_{22}g_1(t) + a_{12}g_2(t) \end{aligned}$$

Bring all terms with g_1 and g_2 to the right side and all the rest to the left side.

$$\boxed{x_1'' - \left(\frac{a_{12}'}{a_{12}} + a_{11} + a_{22}\right)x_1' - \frac{a_{11}'a_{12} - a_{11}a_{12}' + a_{12}(a_{12}a_{21} - a_{11}a_{22})}{a_{12}}x_1 = g_1'(t) - \left(\frac{a_{12}'}{a_{12}} + a_{22}\right)g_1(t) + a_{12}g_2(t)}$$

This is the ODE that x_1 satisfies if a_{11}, a_{12}, a_{21}, and a_{22} are time-dependent (the same procedure can be carried out). Notice that if they are constants, the primed a-terms vanish, and the ODE reduces to the previous boxed result. The initial conditions that x_1 must satisfy are the same as before.