Problem 15

Consider the linear homogeneous system

$$\begin{aligned} x' &= p_{11}(t)x + p_{12}(t)y, \\ y' &= p_{21}(t)x + p_{22}(t)y. \end{aligned}$$

Show that if $x = x_1(t)$, $y = y_1(t)$ and $x = x_2(t)$, $y = y_2(t)$ are two solutions of the given system, then $x = c_1x_1(t) + c_2x_2(t)$, $y = c_1y_1(t) + c_2y_2(t)$ is also a solution for any constants c_1 and c_2. This is the principle of superposition.

Solution

Suppose that $x = x_1(t)$, $y = y_1(t)$ and $x = x_2(t)$, $y = y_2(t)$ are two solutions of the given system.

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + p_{12}(t)y_1 \\ y_1' &= p_{21}(t)x_1 + p_{22}(t)y_1 \end{aligned} \qquad \begin{aligned} x_2' &= p_{11}(t)x_2 + p_{12}(t)y_2 \\ y_2' &= p_{21}(t)x_2 + p_{22}(t)y_2 \end{aligned}$$

Multiply both sides of each equation by c_1 or c_2.

$$\begin{aligned} c_1x_1' &= c_1p_{11}(t)x_1 + c_1p_{12}(t)y_1 \\ c_1y_1' &= c_1p_{21}(t)x_1 + c_1p_{22}(t)y_1 \end{aligned} \qquad \begin{aligned} c_2x_2' &= c_2p_{11}(t)x_2 + c_2p_{12}(t)y_2 \\ c_2y_2' &= c_2p_{21}(t)x_2 + c_2p_{22}(t)y_2 \end{aligned}$$

Add the respective sides of each equation.

$$\begin{aligned} c_1x_1' + c_2x_2' &= [c_1p_{11}(t)x_1 + c_1p_{12}(t)y_1] + [c_2p_{11}(t)x_2 + c_2p_{12}(t)y_2] \\ (c_1x_1 + c_2x_2)' &= p_{11}(t)(c_1x_1 + c_2x_2) + p_{12}(t)(c_1y_1 + c_2y_2) \end{aligned}$$

$$\begin{aligned} c_1y_1' + c_2y_2' &= [c_1p_{21}(t)x_1 + c_1p_{22}(t)y_1] + [c_2p_{21}(t)x_2 + c_2p_{22}(t)y_2] \\ (c_1y_1 + c_2y_2)' &= p_{21}(t)(c_1x_1 + c_2x_2) + p_{22}(t)(c_1y_1 + c_2y_2) \end{aligned}$$

Therefore, $x = c_1x_1 + c_2x_2$ and $y = c_1y_1 + c_2y_2$ are also solutions to the system of ODEs.