

Problem 16

Let $x = x_1(t)$, $y = y_1(t)$ and $x = x_2(t)$, $y = y_2(t)$ be any two solutions of the linear nonhomogeneous system

$$\begin{aligned}x' &= p_{11}(t)x + p_{12}(t)y + g_1(t), \\y' &= p_{21}(t)x + p_{22}(t)y + g_2(t).\end{aligned}$$

Show that $x = x_1(t) - x_2(t)$, $y = y_1(t) - y_2(t)$ is a solution of the corresponding homogeneous system.

Solution

Suppose that $x = x_1(t)$, $y = y_1(t)$ and $x = x_2(t)$, $y = y_2(t)$ are two solutions of the given system.

$$\begin{aligned}x'_1 &= p_{11}(t)x_1 + p_{12}(t)y_1 + g_1(t) & x'_2 &= p_{11}(t)x_2 + p_{12}(t)y_2 + g_1(t) \\y'_1 &= p_{21}(t)x_1 + p_{22}(t)y_1 + g_2(t) & y'_2 &= p_{21}(t)x_2 + p_{22}(t)y_2 + g_2(t)\end{aligned}$$

Subtract the respective sides of each equation.

$$\begin{aligned}x'_1 - x'_2 &= [p_{11}(t)x_1 + p_{12}(t)y_1 + g_1(t)] - [p_{11}(t)x_2 + p_{12}(t)y_2 + g_1(t)] \\(x_1 - x_2)' &= p_{11}(t)(x_1 - x_2) + p_{12}(t)(y_1 - y_2)\end{aligned}$$

$$\begin{aligned}y'_1 - y'_2 &= [p_{21}(t)x_1 + p_{22}(t)y_1 + g_2(t)] - [p_{21}(t)x_2 + p_{22}(t)y_2 + g_2(t)] \\(y_1 - y_2)' &= p_{21}(t)(x_1 - x_2) + p_{22}(t)(y_1 - y_2)\end{aligned}$$

Therefore, $x = x_1 - x_2$ and $y = y_1 - y_2$ are solutions to the corresponding homogeneous system of ODEs,

$$\begin{aligned}x' &= p_{11}(t)x + p_{12}(t)y \\y' &= p_{21}(t)x + p_{22}(t)y.\end{aligned}$$