Problem 19

Consider the circuit shown in Figure 7.1.2. Let I_1, I_2, and I_3 be the currents through the capacitor, resistor, and inductor, respectively. Likewise, let V_1, V_2, and V_3 be the corresponding voltage drops. The arrows denote the arbitrarily chosen directions in which currents and voltage drops will be taken to be positive.

(a) Applying Kirchhoff's second law to the upper loop in the circuit, show that

$$V_1 - V_2 = 0. \tag{i}$$

In a similar way, show that

$$V_2 - V_3 = 0. \tag{ii}$$

(b) Applying Kirchhoff's first law to either node in the circuit, show that

$$I_1 + I_2 + I_3 = 0. \tag{iii}$$

(c) Use the current–voltage relation through each element in the circuit to obtain the equations

$$CV_1' = I_1, \qquad V_2 = RI_2, \qquad LI_3' = V_3. \tag{iv}$$

(d) Eliminate V_2, V_3, I_1, and I_2 among Eqs. (i) through (iv) to obtain

$$CV_1' = -I_3 - \frac{V_1}{R}, \qquad LI_3' = V_1. \tag{v}$$

Observe that if we omit the subscripts in Eqs. (v), then we have the system (2) of this section.

Solution

The RLC circuit being considered here is shown in Figure 7.1.2.

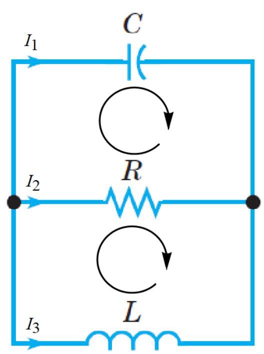


FIGURE 7.1.2 A parallel *LRC* circuit.

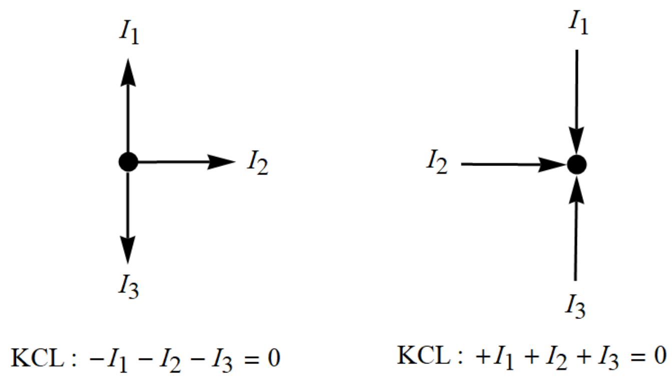
In applying Kirchhoff's voltage law, the two clockwise loops shown above will be used. Because the top loop goes in the direction of I_1 and goes against the direction of I_2, the voltage drops are $+V_1$ and $-V_2$, respectively.

$$\sum V_i = 0 \quad \rightarrow \quad +V_1 - V_2 = 0 \tag{1}$$

Similarly, the bottom loop goes in the direction of I_2 and against the direction of I_3, so the voltage drops are $+V_2$ and $-V_3$, respectively.

$$\sum V_i = 0 \quad \rightarrow \quad +V_2 - V_3 = 0 \tag{2}$$

There are two nodes in the circuit. The currents flow out of the left node and flow into the right node.



Applying Kirchhoff's current law to either one of them results in the same equation.

$$I_1 + I_2 + I_3 = 0 \tag{3}$$

The voltage-current relations for the different circuit elements are as follows.

$$\begin{aligned}
&\text{Capacitor:} && V = \frac{Q}{C} && \Rightarrow && \frac{dV_1}{dt} = \frac{I_1}{C} \\
&\text{Resistor:} && V = IR && \Rightarrow && V_2 = I_2 R \\
&\text{Inductor:} && V = L\frac{dI}{dt} && \Rightarrow && V_3 = L\frac{dI_3}{dt}
\end{aligned}$$

Substitute these first two formulas into equation (3).

$$C\frac{dV_1}{dt} + \frac{V_2}{R} + I_3 = 0$$

From equation (1), $V_1 = V_2$.

$$C\frac{dV_1}{dt} + \frac{V_1}{R} + I_3 = 0$$

Therefore,

$$\boxed{CV_1' = -I_3 - \frac{V_1}{R}.}$$

From equation (1), $V_1 = V_2$, and from equation (2), $V_2 = V_3$. Therefore, the voltage-current relation for the inductor becomes

$$\boxed{LI_3' = V_1.}$$