Problem 20

Consider the circuit shown in Figure 7.1.4. Use the method outlined in Problem 19 to show that the current $I$ through the inductor and the voltage $V$ across the capacitor satisfy the system of differential equations

$$\frac{dI}{dt} = -I - V, \quad \frac{dV}{dt} = 2I - V.$$

![Figure 7.1.4](image)

**FIGURE 7.1.4** The circuit in Problem 20.

Solution

Label the circuit as shown below.

![Labelled circuit](image)

Apply Kirchhoff’s voltage law (KVL) to the two clockwise loops. All currents go in the direction of the arrows except for $I_2$.

Left:

$$\sum V_i = 0 \quad \rightarrow \quad +V_1 - V_2 = 0$$

Right:

$$\sum V_i = 0 \quad \rightarrow \quad +V_2 + V_3 + V_4 = 0$$

Apply Kirchhoff’s current law (KCL) to the three nodes. Whether a current is positive or negative depends if it goes into or out of a node.

$$I_1 + I_2 - I_3 = 0$$
$$I_3 - I_4 = 0$$
$$I_4 - I_2 - I_1 = 0$$
Since \( I_3 = I_4 \), the first and third equations are identical. The current through the inductor and the voltage across the capacitor are the quantities of interest, so \( V_2 = V \) and \( I_4 = I \).

\[
\begin{align*}
V &= V_1 \quad (1) \\
V &= -V_3 - V_4 \quad (2) \\
I &= I_3 \quad (3) \\
I &= I_1 + I_2 \quad (4)
\end{align*}
\]

The voltage-current relations for the different circuit elements are as follows.

- **Capacitor:** \( V = \frac{Q}{C} \Rightarrow \frac{dV_2}{dt} = \frac{I_2}{2} \rightarrow \frac{dV}{dt} = 2I_2 \)
- **Resistor:** \( V = IR \Rightarrow V_1 = I_1(2) \quad V_3 = I_3(1) \rightarrow V_1 = 2I_1 \quad V_3 = I_3 \)
- **Inductor:** \( V = L\frac{dI}{dt} \Rightarrow V_4 = (1)\frac{dI_4}{dt} \rightarrow V_4 = \frac{dI}{dt} \)

There are eight equations and eight unknowns, so this system can be solved. Since \( V_3 = I_3 = I \) and \( V_4 = dI/dt \), equation (2) becomes

\[
V = -I - \frac{dI}{dt},
\]

or

\[
\frac{dI}{dt} = -I - V.
\]

Since \( I_1 = V_1/2 = V/2 \) and \( I_2 = (1/2)dV/dt \), equation (4) becomes

\[
I = \frac{V}{2} + \frac{1}{2} \frac{dV}{dt},
\]

or

\[
\frac{dV}{dt} = 2I - V.
\]