Problem 21

Consider the circuit shown in Figure 7.1.5. Use the method outlined in Problem 19 to show that the current I through the inductor and the voltage V across the capacitor satisfy the system of differential equations

$$L\frac{dI}{dt} = -R_1 I - V, \qquad C\frac{dV}{dt} = I - \frac{V}{R_2}.$$

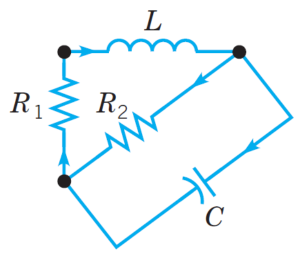
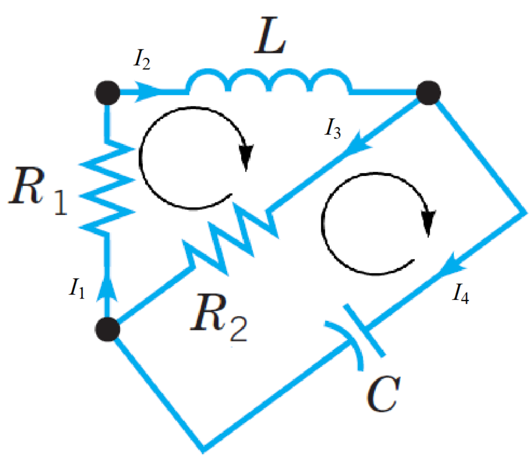


FIGURE 7.1.5 The circuit in Problem 21.

Solution

Label the circuit as shown below.



Apply Kirchhoff's voltage law (KVL) to the two clockwise loops. All currents go in the direction of the arrows except for I_3.

$$\text{Left:} \qquad \sum V_i = 0 \qquad \rightarrow \qquad +V_1 + V_2 + V_3 = 0$$

$$\text{Right:} \qquad \sum V_i = 0 \qquad \rightarrow \qquad -V_3 + V_4 = 0$$

Apply Kirchhoff's current law (KCL) to the three nodes. Whether a current is positive or negative depends if it goes into or out of a node.

$$I_1 - I_2 = 0$$
$$I_3 + I_4 - I_1 = 0$$
$$I_2 - I_3 - I_4 = 0$$

Since $I_1 = I_2$, the second and third equations are identical. The current through the inductor and the voltage across the capacitor are the quantities of interest, so $V_4 = V$ and $I_2 = I$.

$$V = V_3 \tag{1}$$

$$V_3 = -V_1 - V_2 \tag{2}$$

$$I = I_1 \tag{3}$$

$$I = I_3 + I_4 \tag{4}$$

The voltage-current relations for the different circuit elements are as follows.

Capacitor:	$V = \dfrac{Q}{C}$	$\Rightarrow$	$\dfrac{dV_4}{dt} = \dfrac{I_4}{C}$	
Resistor:	$V = IR$	$\Rightarrow$	$V_3 = I_3R_2$	$V_1 = I_1R_1$
Inductor:	$V = L\dfrac{dI}{dt}$	$\Rightarrow$	$V_2 = L\dfrac{dI_2}{dt}$	

With $I = I_2 = I_1$ and these formulas for V_1 and V_2, equation (2) becomes

$$V = -IR_1 - L\frac{dI}{dt},$$

or

$$\boxed{L\frac{dI}{dt} = -R_1I - V.}$$

Since $V_3 = V_4 = V$ and $I_3 = V/R_2$, equation (4) becomes

$$I = \frac{V}{R_2} + C\frac{dV}{dt},$$

or

$$\boxed{C\frac{dV}{dt} = I - \frac{V}{R_2}.}$$