Problem 21

Consider the circuit shown in Figure 7.1.5. Use the method outlined in Problem 19 to show that the current $I$ through the inductor and the voltage $V$ across the capacitor satisfy the system of differential equations

$$
L \frac{dI}{dt} = -R_1 I - V, \quad C \frac{dV}{dt} = I - \frac{V}{R_2}.
$$

**Solution**

Label the circuit as shown below.

Apply Kirchhoff’s voltage law (KVL) to the two clockwise loops. All currents go in the direction of the arrows except for $I_3$.

Left: \[ \sum V_i = 0 \quad \rightarrow \quad + V_1 + V_2 + V_3 = 0 \]

Right: \[ \sum V_i = 0 \quad \rightarrow \quad - V_3 + V_4 = 0 \]

Apply Kirchhoff’s current law (KCL) to the three nodes. Whether a current is positive or negative depends if it goes into or out of a node.

\[
\begin{align*}
I_1 - I_2 &= 0 \\
I_3 + I_4 - I_1 &= 0 \\
I_2 - I_3 - I_4 &= 0
\end{align*}
\]
Since \( I_1 = I_2 \), the second and third equations are identical. The current through the inductor and the voltage across the capacitor are the quantities of interest, so \( V_4 = V \) and \( I_2 = I \).

\[
\begin{align*}
V &= V_3 \\
V_3 &= -V_1 - V_2 \\
I &= I_1 \\
I &= I_3 + I_4
\end{align*}
\]  

(1) \hspace{2cm} (2) \hspace{2cm} (3) \hspace{2cm} (4)

The voltage-current relations for the different circuit elements are as follows.

**Capacitor:** \[ V = \frac{Q}{C} \implies \frac{dV_4}{dt} = \frac{I_4}{C} \]

**Resistor:** \[ V = IR \implies V_3 = I_3R_2 \quad V_1 = I_1R_1 \]

**Inductor:** \[ V = L\frac{dI}{dt} \implies V_2 = L\frac{dI_2}{dt} \]

With \( I = I_2 = I_1 \) and these formulas for \( V_1 \) and \( V_2 \), equation (2) becomes

\[
V = -IR_1 - L\frac{dI}{dt},
\]

or

\[
L\frac{dI}{dt} = -R_1I - V.
\]

Since \( V_3 = V_4 = V \) and \( I_3 = V/R_2 \), equation (4) becomes

\[
I = \frac{V}{R_2} + C\frac{dV}{dt},
\]

or

\[
C\frac{dV}{dt} = I - \frac{V}{R_2}.
\]