Problem 22

Consider the two interconnected tanks shown in Figure 7.1.6. Tank 1 initially contains 30 gal of water and 25 oz of salt, and Tank 2 initially contains 20 gal of water and 15 oz of salt. Water containing 1 oz/gal of salt flows into Tank 1 at a rate of 1.5 gal/min. The mixture flows from Tank 1 to Tank 2 at a rate of 3 gal/min. Water containing 3 oz/gal of salt also flows into Tank 2 at a rate of 1 gal/min (from the outside). The mixture drains from Tank 2 at a rate of 4 gal/min, of which some flows back into Tank 1 at a rate of 1.5 gal/min, while the remainder leaves the system.

(a) Let $Q_1(t)$ and $Q_2(t)$, respectively, be the amount of salt in each tank at time t. Write down differential equations and initial conditions that model the flow process. Observe that the system of differential equations is nonhomogeneous.

(b) Find the values of Q_1 and Q_2 for which the system is in equilibrium—that is, does not change with time. Let Q_1^E and Q_2^E be the equilibrium values. Can you predict which tank will approach its equilibrium state more rapidly?

(c) Let $x_1 = Q_1(t) - Q_1^E$ and $x_2 = Q_2(t) - Q_2^E$. Determine an initial value problem for x_1 and x_2. Observe that the system of equations for x_1 and x_2 is homogeneous.

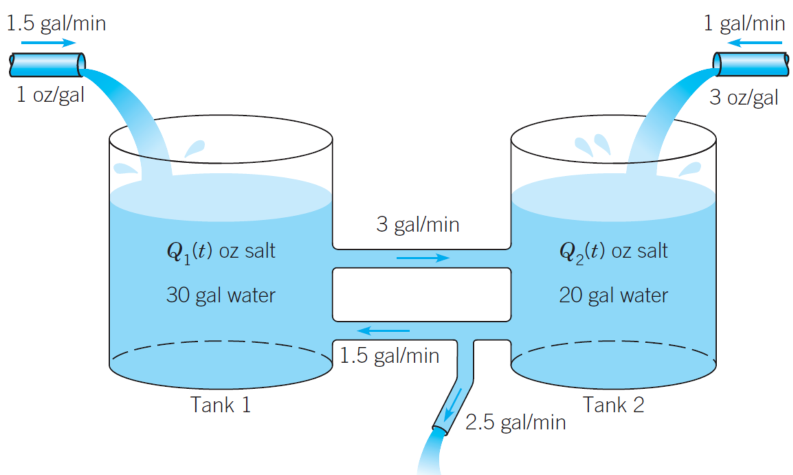

FIGURE 7.1.6 Two interconnected tanks (Problem 22).

Solution

According to the law of conservation of mass, for each tank,

$$\text{Rate of accumulation} = \text{Rate in} - \text{Rate out}.$$

The rate of accumulation is the rate that mass increases with respect to time (dm/dt). The rates in and out are the concentrations in ounces per gallon multiplied by the volumetric flow rates in gallons per minute, assuming everything is perfectly mixed at all times. Let $Q_1(t)$ be the mass in ounces for tank 1, and let $Q_2(t)$ be the mass in ounces for tank 2. Initially, there are 25 ounces and 15 ounces of salt in tank 1 and tank 2, respectively: $Q_1(0) = 25$ and $Q_2(0) = 15$.

Apply the law of conservation of mass to each tank.

$$\frac{dQ_1}{dt} = +1\ \frac{\text{oz}}{\cancel{\text{gal}}} \times 1.5\ \frac{\cancel{\text{gal}}}{\text{min}} + \frac{Q_2(t)}{V_2(t)} \times \frac{1.5\text{ gal}}{\text{min}} - \frac{Q_1(t)}{V_1(t)} \times \frac{3\text{ gal}}{\text{min}}$$

$$\frac{dQ_2}{dt} = +3\ \frac{\text{oz}}{\cancel{\text{gal}}} \times 1\ \frac{\cancel{\text{gal}}}{\text{min}} + \frac{Q_1(t)}{V_1(t)} \times \frac{3\text{ gal}}{\text{min}} - \frac{Q_2(t)}{V_2(t)} \times \frac{4\text{ gal}}{\text{min}}$$

The first terms on the right are the salt solutions pouring in externally, the second terms on the right are from the salt solutions coming in from the opposite tank, and the third terms on the right are from the salt solutions leaving the tank. ODEs for the volume in each tank also need to be formulated.

$$\frac{dV_1}{dt} = +1.5\ \frac{\text{gal}}{\text{min}} - 3\ \frac{\text{gal}}{\text{min}} + 1.5\ \frac{\text{gal}}{\text{min}} = 0$$

$$\frac{dV_2}{dt} = +1\ \frac{\text{gal}}{\text{min}} + 3\ \frac{\text{gal}}{\text{min}} - 1.5\ \frac{\text{gal}}{\text{min}} - 2.5\ \frac{\text{gal}}{\text{min}} = 0$$

These equations imply that the volume in each tank does not change: $V_1(t) = V_1(0) = 30$ gal and $V_2(t) = V_2(0) = 20$ gal. Therefore, the ODEs for the mass become

$$\boxed{\begin{aligned} \frac{dQ_1}{dt} &= 1.5 + 0.075Q_2 - 0.1Q_1, & Q_1(0) &= 25 \\ \frac{dQ_2}{dt} &= 3 + 0.1Q_1 - 0.2Q_2, & Q_2(0) &= 15. \end{aligned}}$$

At equilibrium the mass of salt in each tank is constant; that is, the derivatives are zero.

$$0 = 1.5 + 0.075Q_2^E - 0.1Q_1^E \tag{1}$$

$$0 = 3 + 0.1Q_1^E - 0.2Q_2^E. \tag{2}$$

Solve this system of equations for Q_1^E and Q_2^E.

$$\boxed{Q_1^E = 42 \quad \text{and} \quad Q_2^E = 36}$$

To simplify the boxed ODEs, treat each mass as the sum of a steady component and a transient component.

$$Q_1(t) = Q_1^E + x_1(t)$$

$$Q_2(t) = Q_2^E + x_2(t)$$

Substitute these formulas into the ODEs.

$$\frac{d}{dt}[Q_1^E + x_1(t)] = 1.5 + 0.075[Q_2^E + x_2(t)] - 0.1[Q_1^E + x_1(t)], \qquad Q_1^E + x_1(0) = 25$$

$$\frac{d}{dt}[Q_2^E + x_2(t)] = 3 + 0.1[Q_1^E + x_1(t)] - 0.2[Q_2^E + x_2(t)], \qquad Q_2^E + x_2(0) = 15$$

From equations (1) and (2), the nonhomogeneous terms cancel out.

$$\boxed{\begin{aligned} \frac{dx_1}{dt} &= 0.075x_2 - 0.1x_1, & x_1(0) &= -17 \\ \frac{dx_2}{dt} &= 0.1x_1 - 0.2x_2, & x_2(0) &= -21 \end{aligned}}$$