Problem 23

Consider two interconnected tanks similar to those in Figure 7.1.6. Initially, Tank 1 contains 60 gal of water and Q_1^0 oz of salt, and Tank 2 contains 100 gal of water and Q_2^0 oz of salt. Water containing q_1 oz/gal of salt flows into Tank 1 at a rate of 3 gal/min. The mixture in Tank 1 flows out at a rate of 4 gal/min, of which half flows into Tank 2, while the remainder leaves the system. Water containing q_2 oz/gal of salt also flows into Tank 2 from the outside at the rate of 1 gal/min. The mixture in Tank 2 leaves it at a rate of 3 gal/min, of which some flows back into Tank 1 at a rate of 1 gal/min, while the rest leaves the system.

(a) Draw a diagram that depicts the flow process described above. Let $Q_1(t)$ and $Q_2(t)$, respectively, be the amount of salt in each tank at time t. Write down differential equations and initial conditions for Q_1 and Q_2 that model the flow process.

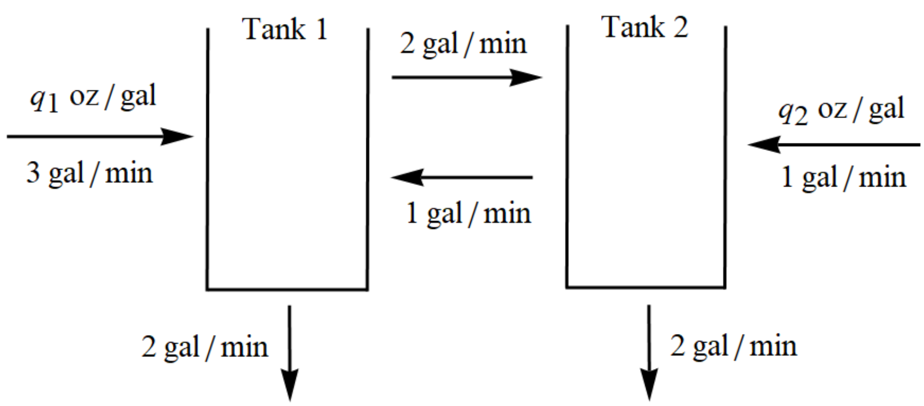
(b) Find the equilibrium values Q_1^E and Q_2^E in terms of the concentrations q_1 and q_2.

(c) Is it possible (by adjusting q_1 and q_2) to obtain $Q_1^E = 60$ and $Q_2^E = 50$ as an equilibrium state?

(d) Describe which equilibrium states are possible for this system for various values of q_1 and q_2.

Solution

Below is a schematic of the two tanks.



According to the law of conservation of mass, for each tank,

$$\text{Rate of accumulation} = \text{Rate in} - \text{Rate out}.$$

The rate of accumulation is the rate that mass increases with respect to time (dm/dt). The rates in and out are the concentrations in ounces per gallon multiplied by the volumetric flow rates in gallons per minute, assuming everything is perfectly mixed at all times. Let $Q_1(t)$ be the mass in ounces for tank 1, and let $Q_2(t)$ be the mass in ounces for tank 2. Initially, there are Q_1^0 ounces and Q_2^0 ounces of salt in tank 1 and tank 2, respectively: $Q_1(0) = Q_1^0$ and $Q_2(0) = Q_2^0$.

Apply the law of conservation of mass to each tank.

$$\frac{dQ_1}{dt} = +q_1 \frac{\text{oz}}{\cancel{\text{gal}}} \times 3 \frac{\cancel{\text{gal}}}{\text{min}} + \frac{Q_2(t)}{V_2(t)} \times \frac{1 \text{ gal}}{\text{min}} - \frac{Q_1(t)}{V_1(t)} \times \frac{4 \text{ gal}}{\text{min}}$$

$$\frac{dQ_2}{dt} = +q_2 \frac{\text{oz}}{\cancel{\text{gal}}} \times 1 \frac{\cancel{\text{gal}}}{\text{min}} + \frac{Q_1(t)}{V_1(t)} \times \frac{2 \text{ gal}}{\text{min}} - \frac{Q_2(t)}{V_2(t)} \times \frac{3 \text{ gal}}{\text{min}}$$

The first terms on the right are the salt solutions pouring in externally, the second terms on the right are from the salt solutions coming in from the opposite tank, and the third terms on the right are from the salt solutions leaving the tank. ODEs for the volume in each tank also need to be formulated.

$$\frac{dV_1}{dt} = +3 \frac{\text{gal}}{\text{min}} - 2 \frac{\text{gal}}{\text{min}} + 1 \frac{\text{gal}}{\text{min}} - 2 \frac{\text{gal}}{\text{min}} = 0$$

$$\frac{dV_2}{dt} = +1 \frac{\text{gal}}{\text{min}} + 2 \frac{\text{gal}}{\text{min}} - 1 \frac{\text{gal}}{\text{min}} - 2 \frac{\text{gal}}{\text{min}} = 0$$

These equations imply that the volume in each tank does not change: $V_1(t) = V_1(0) = 60$ gal and $V_2(t) = V_2(0) = 100$ gal. Therefore, the ODEs for the mass become

$$\boxed{\begin{aligned} \frac{dQ_1}{dt} &= 3q_1 + \frac{1}{100}Q_2 - \frac{1}{15}Q_1, & Q_1(0) &= Q_1^0 \\ \frac{dQ_2}{dt} &= q_2 + \frac{1}{30}Q_1 - \frac{3}{100}Q_2, & Q_2(0) &= Q_2^0. \end{aligned}}$$

At equilibrium the mass of salt in each tank is constant; that is, the derivatives are zero.

$$0 = 3q_1 + \frac{1}{100}Q_2^E - \frac{1}{15}Q_1^E$$

$$0 = q_2 + \frac{1}{30}Q_1^E - \frac{3}{100}Q_2^E$$

Solve this system of equations for Q_1^E and Q_2^E.

$$\boxed{Q_1^E = 6(9q_1 + q_2) \quad \text{and} \quad Q_2^E = 20(3q_1 + 2q_2)}$$

If we set $Q_1^E = 60$ and $Q_2^E = 50$, then the system becomes

$$0 = 3q_1 + \frac{1}{100}(50) - \frac{1}{15}(60)$$

$$0 = q_2 + \frac{1}{30}(60) - \frac{3}{100}(50).$$

Solving it yields

$$q_1 = \frac{7}{6} \quad \text{and} \quad q_2 = -\frac{1}{2},$$

which makes no sense physically—the mass must be a positive number. Therefore, it's not possible to obtain $Q_1^E = 60$ and $Q_2^E = 50$ as an equilibrium state.

Consider the ratio of Q_2^E and Q_1^E.

$$\frac{Q_2^E}{Q_1^E} = \frac{20(3q_1 + 2q_2)}{6(9q_1 + q_2)}$$

As q_1 and q_2 become large, the ratio q_2/q_1 approaches 1.

$$\text{Large } q_1, q_2 : \quad \frac{Q_2^E}{Q_1^E} = \frac{20\left(3 + 2\frac{q_2}{q_1}\right)}{6\left(9 + \frac{q_2}{q_1}\right)} \approx \frac{20(5)}{6(10)} = \frac{5}{3}$$

The bounding cases occur when q_1 and q_2 approach zero separately.

$$\lim_{q_1 \to 0} \frac{Q_2^E}{Q_1^E} = \frac{20(2q_2)}{6(q_2)} = \frac{20}{3}$$

$$\lim_{q_2 \to 0} \frac{Q_2^E}{Q_1^E} = \frac{20(3q_1)}{6(9q_1)} = \frac{10}{9}$$

Therefore,

$$\frac{10}{9} \leq \frac{Q_2^E}{Q_1^E} \leq \frac{20}{3}.$$