Problem 7

Systems of first order equations can sometimes be transformed into a single equation of higher order. Consider the system

$$x_1' = -2x_1 + x_2, \qquad x_2' = x_1 - 2x_2.$$

(a) Solve the first equation for x_2 and substitute into the second equation, thereby obtaining a second order equation for x_1. Solve this equation for x_1 and then determine x_2 also.

(b) Find the solution of the given system that also satisfies the initial conditions $x_1(0) = 2$, $x_2(0) = 3$.

(c) Sketch the curve, for $t \geq 0$, given parametrically by the expressions for x_1 and x_2 obtained in part (b).

Solution

Solving the first equation for x_2 yields

$$x_2 = x_1' + 2x_1 \quad \rightarrow \quad x_2' = x_1'' + 2x_1'.$$

Substitute these formulas into the second equation.

$$\begin{aligned} x_2' &= x_1 - 2x_2 \\ x_1'' + 2x_1' &= x_1 - 2(x_1' + 2x_1) \end{aligned}$$

Bring all terms to the left side.

$$x_1'' + 4x_1' + 3x_1 = 0$$

This is a constant-coefficient linear ODE, so it has solutions of the form $x_1 = e^{rt}$. Determine formulas for the derivatives

$$x_1 = e^{rt} \quad \rightarrow \quad x_1' = re^{rt} \quad \rightarrow \quad x_1'' = r^2 e^{rt}$$

and then plug them into the ODE.

$$r^2 e^{rt} + 4re^{rt} + 3e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^2 + 4r + 3 = 0$$

Solve for r.

$$(r + 3)(r + 1) = 0$$

$$r = \{-3, -1\}$$

Two solutions to the ODE for x_1 are $x_1 = e^{-3t}$ and $x_1 = e^{-t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$x_1(t) = C_1 e^{-3t} + C_2 e^{-t}$$

Take the first derivative.

$$x_1'(t) = -3C_1 e^{-3t} - C_2 e^{-t}$$

Now calculate x_2 using the formula in the beginning.

$$\begin{aligned}x_2 &= x_1' + 2x_1 \\ &= \left(-3C_1e^{-3t} - C_2e^{-t}\right) + 2\left(C_1e^{-3t} + C_2e^{-t}\right) \\ &= -C_1e^{-3t} + C_2e^{-t}\end{aligned}$$

Apply the provided initial conditions to determine C_1 and C_2.

$$\begin{aligned}x_1(0) &= C_1 + C_2 = 2 \\ x_2(0) &= -C_1 + C_2 = 3\end{aligned}$$

Solving this system of equations yields

$$C_1 = -\frac{1}{2} \quad \text{and} \quad C_2 = \frac{5}{2}.$$

Therefore,

$$\begin{aligned}x_1(t) &= -\frac{1}{2}e^{-3t} + \frac{5}{2}e^{-t} \\ x_2(t) &= \frac{1}{2}e^{-3t} + \frac{5}{2}e^{-t}.\end{aligned}$$

Below is a parametric plot of $\{x_1(t), x_2(t)\}$ as t goes from 0 to 50.