Problem 7

Systems of first order equations can sometimes be transformed into a single equation of higher order. Consider the system

\[ \begin{align*}
    x_1' &= -2x_1 + x_2, \\
    x_2' &= x_1 - 2x_2.
\end{align*} \]

(a) Solve the first equation for \(x_2\) and substitute into the second equation, thereby obtaining a second order equation for \(x_1\). Solve this equation for \(x_1\) and then determine \(x_2\) also.

(b) Find the solution of the given system that also satisfies the initial conditions \(x_1(0) = 2\), \(x_2(0) = 3\).

(c) Sketch the curve, for \(t \geq 0\), given parametrically by the expressions for \(x_1\) and \(x_2\) obtained in part (b).

Solution

Solving the first equation for \(x_2\) yields

\[ x_2 = x_1' + 2x_1 \rightarrow x_2' = x_1'' + 2x_1'. \]

Substitute these formulas into the second equation.

\[ \begin{align*}
    x_2' &= x_1 - 2x_2 \\
    x_1'' + 2x_1' &= x_1 - 2(x_1' + 2x_1)
\end{align*} \]

Bring all terms to the left side.

\[ x_1'' + 4x_1' + 3x_1 = 0 \]

This is a constant-coefficient linear ODE, so it has solutions of the form \(x_1 = e^{rt}\). Determine formulas for the derivatives

\[ \begin{align*}
    x_1 &= e^{rt} \rightarrow x_1' = re^{rt} \rightarrow x_1'' = r^2e^{rt}
\end{align*} \]

and then plug them into the ODE.

\[ r^2e^{rt} + 4re^{rt} + 3e^{rt} = 0 \]

Divide both sides by \(e^{rt}\).

\[ r^2 + 4r + 3 = 0 \]

Solve for \(r\).

\[ (r + 3)(r + 1) = 0 \]

\[ r = \{-3, -1\} \]

Two solutions to the ODE for \(x_1\) are \(x_1 = e^{-3t}\) and \(x_1 = e^{-t}\). By the principle of superposition, the general solution is a linear combination of these two.

\[ x_1(t) = C_1e^{-3t} + C_2e^{-t} \]

Take the first derivative.

\[ x_1'(t) = -3C_1e^{-3t} - C_2e^{-t} \]
Now calculate $x_2$ using the formula in the beginning.

$$
x_2 = x'_1 + 2x_1 \\
= (-3C_1e^{-3t} - C_2e^{-t}) + 2(C_1e^{-3t} + C_2e^{-t}) \\
= -C_1e^{-3t} + C_2e^{-t}
$$

Apply the provided initial conditions to determine $C_1$ and $C_2$.

$$
x_1(0) = C_1 + C_2 = 2 \\
x_2(0) = -C_1 + C_2 = 3
$$

Solving this system of equations yields

$$C_1 = -\frac{1}{2} \quad \text{and} \quad C_2 = \frac{5}{2}.$$

Therefore,

$$x_1(t) = -\frac{1}{2}e^{-3t} + \frac{5}{2}e^{-t} \\
x_2(t) = \frac{1}{2}e^{-3t} + \frac{5}{2}e^{-t}.$$

Below is a parametric plot of $\{x_1(t), x_2(t)\}$ as $t$ goes from 0 to 50.