Problem 8

In each of Problems 8 through 12, proceed as in Problem 7.

(a) Transform the given system into a single equation of second order.

(b) Find x_1 and x_2 that also satisfy the given initial conditions.

(c) Sketch the graph of the solution in the x_1x_2-plane for $t \geq 0$.

$$\begin{aligned} x_1' &= 3x_1 - 2x_2, \qquad x_1(0) = 3 \\ x_2' &= 2x_1 - 2x_2, \qquad x_2(0) = \frac{1}{2} \end{aligned}$$

Solution

Solve this first equation for x_2.

$$x_2 = -\frac{1}{2}x_1' + \frac{3}{2}x_1 \quad \rightarrow \quad x_2' = -\frac{1}{2}x_1'' + \frac{3}{2}x_1'$$

Substitute these formulas into the second equation.

$$\begin{aligned} x_2' &= 2x_1 - 2x_2 \\ -\frac{1}{2}x_1'' + \frac{3}{2}x_1' &= 2x_1 - 2\left(-\frac{1}{2}x_1' + \frac{3}{2}x_1\right) \\ -\frac{1}{2}x_1'' + \frac{3}{2}x_1' &= 2x_1 + x_1' - 3x_1 \end{aligned}$$

Bring all terms to the left side and then multiply both sides by -2.

$$x_1'' - x_1' - 2x_1 = 0$$

This is a constant-coefficient linear ODE, so it has solutions of the form $x_1 = e^{rt}$. Determine formulas for the derivatives

$$x_1 = e^{rt} \quad \rightarrow \quad x_1' = re^{rt} \quad \rightarrow \quad x_1'' = r^2e^{rt}$$

and then plug them into the ODE.

$$r^2e^{rt} - re^{rt} - 2e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^2 - r - 2 = 0$$

Solve for r.

$$\begin{gathered} (r+1)(r-2) = 0 \\ r = \{-1, 2\} \end{gathered}$$

Two solutions to the ODE for x_1 are $x_1 = e^{-t}$ and $x_1 = e^{2t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$x_1(t) = C_1e^{-t} + C_2e^{2t}$$

Take the first derivative.

$$x_1'(t) = -C_1e^{-t} + 2C_2e^{2t}$$

Now calculate x_2 using the formula in the beginning.

$$\begin{aligned} x_2 &= -\frac{1}{2}x_1' + \frac{3}{2}x_1 \\ &= -\frac{1}{2}(-C_1e^{-t} + 2C_2e^{2t}) + \frac{3}{2}(C_1e^{-t} + C_2e^{2t}) \\ &= 2C_1e^{-t} + \frac{1}{2}C_2e^{2t} \end{aligned}$$

Apply the provided initial conditions to determine C_1 and C_2.

$$\begin{aligned} x_1(0) &= C_1 + C_2 = 3 \\ x_2(0) &= 2C_1 + \frac{1}{2}C_2 = \frac{1}{2} \end{aligned}$$

Solving this system of equations yields

$$C_1 = -\frac{2}{3} \quad \text{and} \quad C_2 = \frac{11}{3}.$$

Therefore,

$$\begin{aligned} x_1(t) &= -\frac{2}{3}e^{-t} + \frac{11}{3}e^{2t} \\ x_2(t) &= -\frac{4}{3}e^{-t} + \frac{11}{6}e^{2t}. \end{aligned}$$

Below is a parametric plot of $\{x_1(t), x_2(t)\}$ as t goes from 0 to 2.