# Problem 9

In each of Problems 8 through 12, proceed as in Problem 7.

(a) Transform the given system into a single equation of second order.

(b) Find $x_1$ and $x_2$ that also satisfy the given initial conditions.

(c) Sketch the graph of the solution in the $x_1x_2$-plane for $t \geq 0$.

$$\begin{aligned} x_1' &= 1.25x_1 + 0.75x_2, \qquad x_1(0) = -2 \\ x_2' &= 0.75x_1 + 1.25x_2, \qquad x_2(0) = 1 \end{aligned}$$

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**Solution**

Write the system of ODEs using fractions to make the forthcoming calculations easier.

$$\begin{aligned} x_1' &= \frac{5}{4}x_1 + \frac{3}{4}x_2, \qquad x_1(0) = -2 \\ x_2' &= \frac{3}{4}x_1 + \frac{5}{4}x_2, \qquad x_2(0) = 1 \end{aligned}$$

Solve this first equation for $x_2$.

$$x_2 = \frac{4}{3}x_1' - \frac{5}{3}x_1 \quad \rightarrow \quad x_2' = \frac{4}{3}x_1'' - \frac{5}{3}x_1'$$

Substitute these formulas into the second equation.

$$\begin{aligned} x_2' &= \frac{3}{4}x_1 + \frac{5}{4}x_2 \\ \frac{4}{3}x_1'' - \frac{5}{3}x_1' &= \frac{3}{4}x_1 + \frac{5}{4}\left(\frac{4}{3}x_1' - \frac{5}{3}x_1\right) \\ \frac{4}{3}x_1'' - \frac{5}{3}x_1' &= \frac{3}{4}x_1 + \frac{5}{3}x_1' - \frac{25}{12}x_1 \end{aligned}$$

Bring all terms to the left side and then multiply both sides by 3/4.

$$x_1'' - \frac{5}{2}x_1' + x_1 = 0$$

This is a constant-coefficient linear ODE, so it has solutions of the form $x_1 = e^{rt}$. Determine formulas for the derivatives

$$x_1 = e^{rt} \quad \rightarrow \quad x_1' = re^{rt} \quad \rightarrow \quad x_1'' = r^2e^{rt}$$

and then plug them into the ODE.

$$r^2e^{rt} - \frac{5}{2}re^{rt} + e^{rt} = 0$$

Divide both sides by $e^{rt}$ and then solve for $r$.

$$r^2 - \frac{5}{2}r + 1 = 0 \quad \rightarrow \quad \left(r - \frac{1}{2}\right)(r-2) = 0 \quad \rightarrow \quad r = \left\{\frac{1}{2}, 2\right\}$$

Two solutions to the ODE for $x_1$ are $x_1 = e^{t/2}$ and $x_1 = e^{2t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$x_1(t) = C_1 e^{t/2} + C_2 e^{2t}$$

Take the first derivative.

$$x_1'(t) = \frac{C_1}{2} e^{t/2} + 2C_2 e^{2t}$$

Now calculate $x_2$ using the formula in the beginning.

$$\begin{aligned} x_2 &= \frac{4}{3}x_1' - \frac{5}{3}x_1 \\ &= \frac{4}{3}\left(\frac{C_1}{2}e^{t/2} + 2C_2 e^{2t}\right) - \frac{5}{3}(C_1 e^{t/2} + C_2 e^{2t}) \\ &= -C_1 e^{t/2} + C_2 e^{2t} \end{aligned}$$

Apply the provided initial conditions to determine $C_1$ and $C_2$.

$$\begin{aligned} x_1(0) &= C_1 + C_2 = -2 \\ x_2(0) &= -C_1 + C_2 = 1 \end{aligned}$$

Solving this system of equations yields

$$C_1 = -\frac{3}{2} \quad \text{and} \quad C_2 = -\frac{1}{2}.$$

Therefore,

$$x_1(t) = -\frac{3}{2}e^{t/2} - \frac{1}{2}e^{2t}$$

$$x_2(t) = \frac{3}{2}e^{t/2} - \frac{1}{2}e^{2t}.$$

Below is a parametric plot of $\{x_1(t), x_2(t)\}$ as $t$ goes from 0 to 2.