Problem 9

In each of Problems 8 through 12, proceed as in Problem 7.

(a) Transform the given system into a single equation of second order.

(b) Find $x_1$ and $x_2$ that also satisfy the given initial conditions.

(c) Sketch the graph of the solution in the $x_1x_2$-plane for $t \geq 0$.

$x_1' = 1.25x_1 + 0.75x_2, \quad x_1(0) = -2$

$x_2' = 0.75x_1 + 1.25x_2, \quad x_2(0) = 1$

Solution

Write the system of ODEs using fractions to make the forthcoming calculations easier.

$x_1' = \frac{5}{4}x_1 + \frac{3}{4}x_2, \quad x_1(0) = -2$

$x_2' = \frac{3}{4}x_1 + \frac{5}{4}x_2, \quad x_2(0) = 1$

Solve this first equation for $x_2$.

$x_2 = \frac{4}{3}x_1' - \frac{5}{3}x_1 \quad \rightarrow \quad x_2 = \frac{4}{3}x_1'' - \frac{5}{3}x_1'$

Substitute these formulas into the second equation.

$x_2' = \frac{3}{4}x_1 + \frac{5}{4}x_2$

$\frac{4}{3}x_1'' - \frac{5}{3}x_1' = \frac{3}{4}x_1 + \frac{5}{4}\left(\frac{4}{3}x_1'' - \frac{5}{3}x_1\right)$

$\frac{4}{3}x_1'' - \frac{5}{3}x_1' = \frac{3}{4}x_1 + \frac{5}{3}x_1' - \frac{25}{12}x_1$

Bring all terms to the left side and then multiply both sides by $3/4$.

$x_1'' - \frac{5}{2}x_1' + x_1 = 0$

This is a constant-coefficient linear ODE, so it has solutions of the form $x_1 = e^{rt}$. Determine formulas for the derivatives

$x_1 = e^{rt} \quad \rightarrow \quad x_1' = re^{rt} \quad \rightarrow \quad x_1'' = r^2e^{rt}$

and then plug them into the ODE.

$r^2e^{rt} - \frac{5}{2}re^{rt} + e^{rt} = 0$

Divide both sides by $e^{rt}$ and then solve for $r$.

$r^2 - \frac{5}{2}r + 1 = 0 \quad \rightarrow \quad \left(r - \frac{1}{2}\right)(r - 2) = 0 \quad \rightarrow \quad r = \left\{\frac{1}{2}, 2\right\}$

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Two solutions to the ODE for $x_1$ are $x_1 = e^{t/2}$ and $x_1 = e^{2t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$x_1(t) = C_1 e^{t/2} + C_2 e^{2t}$$

Take the first derivative.

$$x_1'(t) = \frac{C_1}{2} e^{t/2} + 2C_2 e^{2t}$$

Now calculate $x_2$ using the formula in the beginning.

$$x_2 = \frac{4}{3} x_1' - \frac{5}{3} x_1$$

$$= \frac{4}{3} \left( \frac{C_1}{2} e^{t/2} + 2C_2 e^{2t} \right) - \frac{5}{3} \left( C_1 e^{t/2} + C_2 e^{2t} \right)$$

$$= -C_1 e^{t/2} + C_2 e^{2t}$$

Apply the provided initial conditions to determine $C_1$ and $C_2$.

$$x_1(0) = C_1 + C_2 = -2$$

$$x_2(0) = -C_1 + C_2 = 1$$

Solving this system of equations yields

$$C_1 = -\frac{3}{2} \quad \text{and} \quad C_2 = -\frac{1}{2}$$

Therefore,

$$x_1(t) = -\frac{3}{2} e^{t/2} - \frac{1}{2} e^{2t}$$

$$x_2(t) = \frac{3}{2} e^{t/2} - \frac{1}{2} e^{2t}.$$ 

Below is a parametric plot of $\{x_1(t), x_2(t)\}$ as $t$ goes from 0 to 2.