Problem 13

In each of Problems 10 through 19, either compute the inverse of the given matrix, or else show that it is singular.

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Solution

Start by calculating the determinant.

$$\det\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix} = 1\begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 1(-2-1) - 1(4-1) - 1(2+1) = -9$$

Since it's not zero, an inverse for the given matrix exists.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right)$$

The aim is to make the left side 1's and 0's as the right side is now. The top left entry is already 1. To make the mid-left entry 0, multiply the first row by -2 and add it to the second row.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -3 & 3 & -2 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right)$$

Multiply the first row by -1 and add it to the third row to make the bottom left entry 0.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -3 & 3 & -2 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 & 1 \end{array}\right)$$

Multiply the third row by -1 and add it to the second row to make the middle right entry 0.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -3 & 0 & -1 & 1 & -1 \\ 0 & 0 & 3 & -1 & 0 & 1 \end{array}\right)$$

Divide the third row by 3.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -3 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} \end{array}\right)$$

Divide the second row by -3.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} \end{array}\right)$$

Add the third row to the first row to make the top right entry 0.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} \end{array}\right)$$

Multiply the second row by -1 and add it to the first row to make the top middle entry 0.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} \end{array}\right)$$

Therefore, the inverse of the given matrix is

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}.$$