Problem 16

In each of Problems 10 through 19, either compute the inverse of the given matrix, or else show that it is singular.

$$\begin{pmatrix} 1 & -1 & -1 \\ 2 & 1 & 0 \\ 3 & -2 & 1 \end{pmatrix}$$

Solution

Start by calculating the determinant.

$$\det\begin{pmatrix} 1 & -1 & -1 \\ 2 & 1 & 0 \\ 3 & -2 & 1 \end{pmatrix} = 1\begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} = 1(1-0) + 1(2-0) - 1(-4-3) = 10$$

Since it's not zero, an inverse for the given matrix exists.

$$\left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & -2 & 1 & 0 & 0 & 1 \end{array}\right)$$

The aim is to make the left side 1's and 0's as the right side is now. Multiply the first row by -2 and add it to the second row to make the middle left entry 0.

$$\left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 3 & 2 & -2 & 1 & 0 \\ 3 & -2 & 1 & 0 & 0 & 1 \end{array}\right)$$

Multiply the first row by -3 and add it to the third row to make the bottom left entry 0.

$$\left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 3 & 2 & -2 & 1 & 0 \\ 0 & 1 & 4 & -3 & 0 & 1 \end{array}\right)$$

Multiply the second row by -2 and add it to the third row to make the bottom right entry 0.

$$\left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 3 & 2 & -2 & 1 & 0 \\ 0 & -5 & 0 & 1 & -2 & 1 \end{array}\right)$$

Switch the second row with the third row.

$$\left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & -5 & 0 & 1 & -2 & 1 \\ 0 & 3 & 2 & -2 & 1 & 0 \end{array}\right)$$

Divide the second row by -5.

$$\left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ 0 & 3 & 2 & -2 & 1 & 0 \end{array}\right)$$

Add the second row to the first row to make the top middle entry 0.

$$\left(\begin{array}{ccr|ccc} 1 & 0 & -1 & \frac{4}{5} & \frac{2}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ 0 & 3 & 2 & -2 & 1 & 0 \end{array}\right)$$

Multiply the second row by -3 and add it to the third row to make the bottom middle entry 0.

$$\left(\begin{array}{ccr|ccc} 1 & 0 & -1 & \frac{4}{5} & \frac{2}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 2 & -\frac{7}{5} & -\frac{1}{5} & \frac{3}{5} \end{array}\right)$$

Divide the third row by 2.

$$\left(\begin{array}{ccr|ccc} 1 & 0 & -1 & \frac{4}{5} & \frac{2}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & -\frac{7}{10} & -\frac{1}{10} & \frac{3}{10} \end{array}\right)$$

Add the third row to the first row.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{10} & \frac{3}{10} & \frac{1}{10} \\ 0 & 1 & 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & -\frac{7}{10} & -\frac{1}{10} & \frac{3}{10} \end{array}\right)$$

Therefore, the inverse of the given matrix is

$$\begin{pmatrix} \frac{1}{10} & \frac{3}{10} & \frac{1}{10} \\ -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ -\frac{7}{10} & -\frac{1}{10} & \frac{3}{10} \end{pmatrix}.$$