Problem 18

In each of Problems 10 through 19, either compute the inverse of the given matrix, or else show that it is singular.

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$

Solution

Start by calculating the determinant.

$$\begin{aligned} \det \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix} &= 1 \begin{vmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} - 0 \begin{vmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} + 0 \begin{vmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} \\ &= 1 \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} \\ &= 1(-1-0) - (1-1) \\ &= -1 \end{aligned}$$

Since it's not zero, an inverse for the given matrix exists.

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

The aim is to make the left side 1's and 0's as the right side is now. Add the second row to the fourth row to make the bottom two entries 0.

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right)$$

Add the fourth row to the first row to make the top right entry 0.

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right)$$

Add the first row to the third row.

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right)$$

Multiply the third row by -1 and add it to the second row.

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right)$$

Multiply the second row by -1.

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right)$$

Therefore, the inverse for the given matrix is

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$