Problem 20

If $\mathbf{A}$ is a square matrix, and if there are two matrices $\mathbf{B}$ and $\mathbf{C}$ such that $\mathbf{AB} = \mathbf{I}$ and $\mathbf{CA} = \mathbf{I}$, show that $\mathbf{B} = \mathbf{C}$. Thus, if a matrix has an inverse, it can have only one.

Solution

Pre-multiply both sides of $\mathbf{AB} = \mathbf{I}$ by $\mathbf{A}^{-1}$.

$$\mathbf{A}^{-1}\mathbf{AB} = \mathbf{A}^{-1}\mathbf{I}$$

$$(\mathbf{A}^{-1}\mathbf{A})\mathbf{B} = \mathbf{A}^{-1}$$

$$\mathbf{IB} = \mathbf{A}^{-1}$$

$$\mathbf{B} = \mathbf{A}^{-1}$$

Post-multiply both sides of $\mathbf{CA} = \mathbf{I}$ by $\mathbf{A}^{-1}$.

$$\mathbf{CAA}^{-1} = \mathbf{IA}^{-1}$$

$$\mathbf{C}(\mathbf{AA}^{-1}) = \mathbf{A}^{-1}$$

$$\mathbf{CI} = \mathbf{A}^{-1}$$

$$\mathbf{C} = \mathbf{A}^{-1}$$

Therefore, $\mathbf{B} = \mathbf{C}$.