Problem 6

If $\mathbf{A} = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ -2 & 0 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 1 & -1 \\ -2 & 3 & 3 \\ 1 & 0 & 2 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 1 & -1 \end{pmatrix}$, verify that

(a) $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ (b) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$

(c) $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$

Solution

$$\begin{aligned}
\mathbf{AB} &= \begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ -2 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 \\ -2 & 3 & 3 \\ 1 & 0 & 2 \end{pmatrix} \\
&= \begin{pmatrix} (1)(2) + (-2)(-2) + (0)(1) & (1)(1) + (-2)(3) + (0)(0) & (1)(-1) + (-2)(3) + (0)(2) \\ (3)(2) + (2)(-2) + (-1)(1) & (3)(1) + (2)(3) + (-1)(0) & (3)(-1) + (2)(3) + (-1)(2) \\ (-2)(2) + (0)(-2) + (3)(1) & (-2)(1) + (0)(3) + (3)(0) & (-2)(-1) + (0)(3) + (3)(2) \end{pmatrix} \\
&= \begin{pmatrix} 6 & -5 & -7 \\ 1 & 9 & 1 \\ -1 & -2 & 8 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
(\mathbf{AB})\mathbf{C} &= \begin{pmatrix} 6 & -5 & -7 \\ 1 & 9 & 1 \\ -1 & -2 & 8 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 7 & -11 & -3 \\ 11 & 20 & 17 \\ -4 & 3 & -12 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{BC} &= \begin{pmatrix} 2 & 1 & -1 \\ -2 & 3 & 3 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 5 & 3 & 3 \\ -1 & 7 & 3 \\ 2 & 3 & -2 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{A}(\mathbf{BC}) &= \begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ -2 & 0 & 3 \end{pmatrix} \begin{pmatrix} 5 & 3 & 3 \\ -1 & 7 & 3 \\ 2 & 3 & -2 \end{pmatrix} \\
&= \begin{pmatrix} 7 & -11 & -3 \\ 11 & 20 & 17 \\ -4 & 3 & -12 \end{pmatrix}
\end{aligned}$$

Therefore, $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

$$\mathbf{A}+\mathbf{B} = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ -2 & 0 & 3 \end{pmatrix} + \begin{pmatrix} 2 & 1 & -1 \\ -2 & 3 & 3 \\ 1 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -1 & -1 \\ 1 & 5 & 2 \\ -1 & 0 & 5 \end{pmatrix}$$

$$(\mathbf{A}+\mathbf{B})+\mathbf{C} = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 5 & 2 \\ -1 & 0 & 5 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 & -1 \\ 2 & 7 & 4 \\ -1 & 1 & 4 \end{pmatrix}$$

$$\mathbf{B}+\mathbf{C} = \begin{pmatrix} 2 & 1 & -1 \\ -2 & 3 & 3 \\ 1 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 2 & -1 \\ -1 & 5 & 5 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\mathbf{A}+(\mathbf{B}+\mathbf{C}) = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ -2 & 0 & 3 \end{pmatrix} + \begin{pmatrix} 4 & 2 & -1 \\ -1 & 5 & 5 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 & -1 \\ 2 & 7 & 4 \\ -1 & 1 & 4 \end{pmatrix}$$

Therefore, $(\mathbf{A}+\mathbf{B})+\mathbf{C} = \mathbf{A}+(\mathbf{B}+\mathbf{C})$. And as shown below, $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$.

$$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ -2 & 0 & 3 \end{pmatrix} \begin{pmatrix} 4 & 2 & -1 \\ -1 & 5 & 5 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & -8 & -11 \\ 9 & 15 & 6 \\ -5 & -1 & 5 \end{pmatrix}$$

$$\mathbf{AB}+\mathbf{AC} = \begin{pmatrix} 6 & -5 & -7 \\ 1 & 9 & 1 \\ -1 & -2 & 8 \end{pmatrix} + \begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ -2 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 6 & -5 & -7 \\ 1 & 9 & 1 \\ -1 & -2 & 8 \end{pmatrix} + \begin{pmatrix} 0 & -3 & -4 \\ 8 & 6 & 5 \\ -4 & 1 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & -8 & -11 \\ 9 & 15 & 6 \\ -5 & -1 & 5 \end{pmatrix}$$