Problem 7

Prove each of the following laws of matrix algebra:

(a) $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$ (b) $\mathbf{A}+(\mathbf{B}+\mathbf{C})=(\mathbf{A}+\mathbf{B})+\mathbf{C}$

(c) $\alpha(\mathbf{A}+\mathbf{B})=\alpha\mathbf{A}+\alpha\mathbf{B}$ (d) $(\alpha+\beta)\mathbf{A}=\alpha\mathbf{A}+\beta\mathbf{A}$

(e) $\mathbf{A}(\mathbf{B}\mathbf{C})=(\mathbf{A}\mathbf{B})\mathbf{C}$ (f) $\mathbf{A}(\mathbf{B}+\mathbf{C})=\mathbf{A}\mathbf{B}+\mathbf{A}\mathbf{C}$

Solution

The first law follows from the commutative law of addition.

$$\begin{aligned}\mathbf{A}+\mathbf{B}&=(a_{ij})+(b_{ij})\\&=(a_{ij}+b_{ij})\\&=(b_{ij}+a_{ij})\\&=(b_{ij})+(a_{ij})\\&=\mathbf{B}+\mathbf{A}\end{aligned}$$

The second law follows from the associative law of addition.

$$\begin{aligned}\mathbf{A}+(\mathbf{B}+\mathbf{C})&=(a_{ij})+[(b_{ij})+(c_{ij})]\\&=(a_{ij})+(b_{ij}+c_{ij})\\&=(a_{ij}+b_{ij}+c_{ij})\\&=(a_{ij}+b_{ij})+(c_{ij})\\&=[(a_{ij})+(b_{ij})]+(c_{ij})\\&=(\mathbf{A}+\mathbf{B})+\mathbf{C}\end{aligned}$$

The third law follows from the distributive law.

$$\begin{aligned}\alpha(\mathbf{A}+\mathbf{B})&=\alpha[(a_{ij})+(b_{ij})]\\&=\alpha(a_{ij}+b_{ij})\\&=(\alpha(a_{ij}+b_{ij}))\\&=(\alpha a_{ij}+\alpha b_{ij})\\&=(\alpha a_{ij})+(\alpha b_{ij})\\&=\alpha(a_{ij})+\alpha(b_{ij})\\&=\alpha\mathbf{A}+\alpha\mathbf{B}\end{aligned}$$

The fourth law follows from the distributive law.

$$\begin{aligned}(\alpha+\beta)\mathbf{A} &= (\alpha+\beta)(a_{ij}) \\ &= ((\alpha+\beta)a_{ij}) \\ &= (\alpha a_{ij}+\beta a_{ij}) \\ &= (\alpha a_{ij})+(\beta a_{ij}) \\ &= \alpha(a_{ij})+\beta(a_{ij}) \\ &= \alpha\mathbf{A}+\beta\mathbf{A}\end{aligned}$$

Use the formula for the multiplication of matrices twice for each side of the fifth law.

$$\begin{aligned}\mathbf{A}(\mathbf{BC}) &= (a_{ij})\left(\sum_{k=1}^{n} b_{ik}c_{kj}\right) \\ &= \left(\sum_{l=1}^{n} a_{il}\sum_{k=1}^{n} b_{lk}c_{kj}\right) \\ &= \left(\sum_{k=1}^{n}\sum_{l=1}^{n} a_{il}b_{lk}c_{kj}\right)\end{aligned}$$

k and l are dummy indices, so let l be k and let k be l.

$$\begin{aligned}\mathbf{A}(\mathbf{BC}) &= \left(\sum_{l=1}^{n}\sum_{k=1}^{n} a_{ik}b_{kl}c_{lj}\right) \\ &= \left(\sum_{k=1}^{n} a_{ik}b_{kj}\right)(c_{ij}) \\ &= (\mathbf{AB})\mathbf{C}\end{aligned}$$

Use the formula for the multiplication of matrices to establish the sixth law.

$$\begin{aligned}\mathbf{A}(\mathbf{B}+\mathbf{C}) &= (a_{ij})[(b_{ij})+(c_{ij})] \\ &= (a_{ij})(b_{ij}+c_{ij}) \\ &= \sum_{k=1}^{n} a_{ik}(b_{kj}+c_{kj}) \\ &= \sum_{k=1}^{n}(a_{ik}b_{kj}+a_{ik}c_{kj}) \\ &= \left(\sum_{k=1}^{n} a_{ik}b_{kj}\right)+\left(\sum_{k=1}^{n} a_{ik}c_{kj}\right) \\ &= \mathbf{AB}+\mathbf{AC}\end{aligned}$$