Problem 1

In each of Problems 1 through 6, either solve the given system of equations, or else show that there is no solution.

$$\begin{aligned} x_1 \quad\;\; - \;\; x_3 &= 0 \\ 3x_1 + x_2 + \;\; x_3 &= 1 \\ -x_1 + x_2 + 2x_3 &= 2 \end{aligned}$$

Solution

Start by calculating the determinant of the coefficient matrix.

$$\det \begin{pmatrix} 1 & 0 & -1 \\ 3 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} = 1 \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} - 1 \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} = 1(2-1) - 1(3+1) = -3$$

Since it's not zero, a unique solution to the system of equations exists. Solve the first equation for x_3.

$$x_3 = x_1$$

Substitute this result into the other two equations.

$$\begin{aligned} 3x_1 + x_2 + (x_1) &= 1 \\ -x_1 + x_2 + 2(x_1) &= 2 \end{aligned}$$

Simplify the left sides.

$$\begin{aligned} 4x_1 + x_2 &= 1 \\ x_1 + x_2 &= 2 \end{aligned}$$

Solve the second equation for x_1,

$$x_1 = 2 - x_2,$$

and substitute it into the first equation.

$$4(2 - x_2) + x_2 = 1$$

Solve for x_2.

$$x_2 = \frac{7}{3}$$

That means

$$x_3 = x_1 = 2 - \left(\frac{7}{3}\right) = -\frac{1}{3}.$$

Therefore,

$$\left\{-\frac{1}{3}, \frac{7}{3}, -\frac{1}{3}\right\}.$$