Problem 10

In each of Problems 7 through 11, determine whether the members of the given set of vectors are linearly independent. If they are linearly dependent, find a linear relation among them. The vectors are written as row vectors to save space but may be considered as column vectors; that is, the transposes of the given vectors may be used instead of the vectors themselves.

$$\mathbf{x}^{(1)} = (1, 2, -1, 0), \qquad \mathbf{x}^{(2)} = (2, 3, 1, -1), \qquad \mathbf{x}^{(3)} = (-1, 0, 2, 2), \qquad \mathbf{x}^{(4)} = (3, -1, 1, 3)$$

Solution

The four vectors are linearly dependent if there exists a nontrivial solution to

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} + c_4\mathbf{x}^{(4)} = \mathbf{0}$$

for c_1, c_2, c_3, and c_4. Rewrite this equation.

$$c_1\begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 3 \\ 1 \\ -1 \end{pmatrix} + c_3\begin{pmatrix} -1 \\ 0 \\ 2 \\ 2 \end{pmatrix} + c_4\begin{pmatrix} 3 \\ -1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 3 & 0 & -1 \\ -1 & 1 & 2 & 1 \\ 0 & -1 & 2 & 3 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Calculate the determinant of the coefficient matrix.

$$\begin{aligned} \det\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 3 & 0 & -1 \\ -1 & 1 & 2 & 1 \\ 0 & -1 & 2 & 3 \end{pmatrix} &= -2\begin{vmatrix} 2 & -1 & 3 \\ 1 & 2 & 1 \\ -1 & 2 & 3 \end{vmatrix} + 3\begin{vmatrix} 1 & -1 & 3 \\ -1 & 2 & 1 \\ 0 & 2 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 0 & -1 & 2 \end{vmatrix} \\ &= -70 \end{aligned}$$

Since it's not zero, there is a unique solution for c_1, c_2, c_3, and c_4.

$$\begin{aligned} c_1 + 2c_2 - c_3 + 3c_4 &= 0 \\ 2c_1 + 3c_2 - c_4 &= 0 \\ -c_1 + c_2 + 2c_3 + c_4 &= 0 \\ -c_2 + 2c_3 + 3c_4 &= 0 \end{aligned}$$

By inspection, $c_1 = c_2 = c_3 = c_4 = 0$ is the (trivial) solution. Therefore, the four given vectors are linearly independent.