Problem 11

In each of Problems 7 through 11, determine whether the members of the given set of vectors are linearly independent. If they are linearly dependent, find a linear relation among them. The vectors are written as row vectors to save space but may be considered as column vectors; that is, the transposes of the given vectors may be used instead of the vectors themselves.

$$\mathbf{x}^{(1)} = (1, 2, -2), \qquad \mathbf{x}^{(2)} = (3, 1, 0), \qquad \mathbf{x}^{(3)} = (2, -1, 1), \qquad \mathbf{x}^{(4)} = (4, 3, -2)$$

Solution

The four vectors are linearly dependent if there exists a nontrivial solution to

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} + c_4\mathbf{x}^{(4)} = \mathbf{0}$$

for c_1, c_2, c_3, and c_4. Rewrite this equation.

$$c_1\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} + c_2\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + c_4\begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 2 & 4 \\ 2 & 1 & -1 & 3 \\ -2 & 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solve the implied system of equations.

$$\left(\begin{array}{cccc|c} 1 & 3 & 2 & 4 & 0 \\ 2 & 1 & -1 & 3 & 0 \\ -2 & 0 & 1 & -2 & 0 \end{array}\right)$$

Multiply the first row by -2 and add it to the second row.

$$\left(\begin{array}{cccc|c} 1 & 3 & 2 & 4 & 0 \\ 0 & -5 & -5 & -5 & 0 \\ -2 & 0 & 1 & -2 & 0 \end{array}\right)$$

Multiply the first row by 2 and add it to the third row.

$$\left(\begin{array}{cccc|c} 1 & 3 & 2 & 4 & 0 \\ 0 & -5 & -5 & -5 & 0 \\ 0 & 6 & 5 & 6 & 0 \end{array}\right)$$

Add the third row to the second row.

$$\left(\begin{array}{cccc|c} 1 & 3 & 2 & 4 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 6 & 5 & 6 & 0 \end{array}\right)$$

Multiply the second row by -6 and add it to the third row.

$$\left(\begin{array}{cccc|c} 1 & 3 & 2 & 4 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 \end{array}\right)$$

Divide the third row by 5.

$$\left(\begin{array}{cccc|c} 1 & 3 & 2 & 4 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

As a result, the coefficients satisfy the following equations.

$$\begin{aligned} c_1 + 3c_2 + 2c_3 + 4c_4 &= 0 \\ c_2 + c_4 &= 0 \\ c_3 &= 0 \end{aligned}$$

Set $c_3 = 0$ and $c_4 = -c_2$ in the first equation.

$$c_1 + 3c_2 + 2(0) + 4(-c_2) = 0$$

Solve for c_1.

$$c_1 = c_2$$

The solution to the system in terms of the free variable c_2 is then

$$\{c_2, c_2, 0, -c_2\}.$$

If $c_2 = 1$, for example, then

$$\mathbf{x}^{(1)} + \mathbf{x}^{(2)} - \mathbf{x}^{(4)} = \mathbf{0}.$$

Therefore, the four given vectors are linearly dependent.