Problem 12

Suppose that each of the vectors $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}$ has n components, where $n < m$.
Show that $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}$ are linearly dependent.

Solution

The m vectors are linearly dependent if there exists a nontrivial solution to

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \cdots + c_{m-1}\mathbf{x}^{(m-1)} + c_m\mathbf{x}^{(m)} = \mathbf{0}$$

for $c_1, c_2, \ldots, c_{m-1}$, and c_m. Rewrite this vector equation.

$$c_1\begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ \vdots \\ x_{n-1}^{(1)} \\ x_n^{(1)} \end{pmatrix} + c_2\begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \\ \vdots \\ x_{n-1}^{(2)} \\ x_n^{(2)} \end{pmatrix} + \cdots + c_{m-1}\begin{pmatrix} x_1^{(m-1)} \\ x_2^{(m-1)} \\ \vdots \\ x_{n-1}^{(m-1)} \\ x_n^{(m-1)} \end{pmatrix} + c_m\begin{pmatrix} x_1^{(m)} \\ x_2^{(m)} \\ \vdots \\ x_{n-1}^{(m)} \\ x_n^{(m)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

The implied system of equations is as follows.

$$\begin{aligned} c_1x_1^{(1)} + c_2x_1^{(2)} + \cdots + c_{m-1}x_1^{(m-1)} + c_mx_1^{(m)} &= 0 \\ c_1x_2^{(1)} + c_2x_2^{(2)} + \cdots + c_{m-1}x_2^{(m-1)} + c_mx_2^{(m)} &= 0 \\ &\vdots \\ c_1x_{n-1}^{(1)} + c_2x_{n-1}^{(2)} + \cdots + c_{m-1}x_{n-1}^{(m-1)} + c_mx_{n-1}^{(m)} &= 0 \\ c_1x_n^{(1)} + c_2x_n^{(2)} + \cdots + c_{m-1}x_n^{(m-1)} + c_mx_n^{(m)} &= 0 \end{aligned}$$

Because there are more variables than equations ($m > n$), there are infinitely many solutions for $c_1, c_2, \ldots, c_{m-1}$, and c_m. Therefore, the m vectors are linearly dependent.