Problem 13

In each of Problems 13 and 14, determine whether the members of the given set of vectors are linearly independent for $-\infty < t < \infty$. If they are linearly dependent, find the linear relation among them. As in Problems 7 through 11, the vectors are written as row vectors to save space.

$$\mathbf{x}^{(1)}(t) = (e^{-t}, 2e^{-t}), \qquad \mathbf{x}^{(2)}(t) = (e^{-t}, e^{-t}), \qquad \mathbf{x}^{(3)}(t) = (3e^{-t}, 0)$$

Solution

The three vectors are linearly dependent if there exists a nontrivial solution to

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} = \mathbf{0}$$

for c_1, c_2, and c_3. Rewrite this vector equation.

$$c_1\begin{pmatrix} e^{-t} \\ 2e^{-t} \end{pmatrix} + c_2\begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix} + c_3\begin{pmatrix} 3e^{-t} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The implied system of equations is as follows.

$$\begin{aligned} c_1e^{-t} + c_2e^{-t} + 3c_3e^{-t} &= 0 \\ 2c_1e^{-t} + c_2e^{-t} &= 0 \end{aligned}$$

Multiply both sides of each equation by e^t.

$$\begin{aligned} c_1 + c_2 + 3c_3 &= 0 \\ 2c_1 + c_2 &= 0 \end{aligned}$$

Solve this second equation for c_2

$$c_2 = -2c_1$$

and plug it into the first equation.

$$c_1 + (-2c_1) + 3c_3 = 0$$

Solve for c_1.

$$c_1 = 3c_3$$

Therefore, in terms of the free variable c_3, the solution to the system is

$$\{3c_3, -6c_3, c_3\}.$$

If $c_3 = 1$, for example, then

$$3\mathbf{x}^{(1)} - 6\mathbf{x}^{(2)} + \mathbf{x}^{(3)} = \mathbf{0}.$$

Therefore, the given three vectors are linearly dependent for $-\infty < t < \infty$.