# Problem 14

In each of Problems 13 and 14, determine whether the members of the given set of vectors are linearly independent for $-\infty < t < \infty$. If they are linearly dependent, find the linear relation among them. As in Problems 7 through 11, the vectors are written as row vectors to save space.

$$\mathbf{x}^{(1)}(t) = (2\sin t, \sin t), \qquad \mathbf{x}^{(2)}(t) = (\sin t, 2\sin t)$$

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**Solution**

The two vectors are linearly dependent if there exists a nontrivial solution to

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} = \mathbf{0}$$

for $c_1$ and $c_2$. Rewrite this vector equation.

$$c_1\begin{pmatrix} 2\sin t \\ \sin t \end{pmatrix} + c_2\begin{pmatrix} \sin t \\ 2\sin t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2\sin t & \sin t \\ \sin t & 2\sin t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Calculate the determinant of the coefficient matrix.

$$\det\begin{pmatrix} 2\sin t & \sin t \\ \sin t & 2\sin t \end{pmatrix} = 4\sin^2 t - \sin^2 t = 3\sin^2 t$$

Since it's not zero, there is a unique solution for $c_1$ and $c_2$.

$$c_1(2\sin t) + c_2(\sin t) = 0$$
$$c_1(\sin t) + c_2(2\sin t) = 0$$

By inspection, $c_1 = c_2 = 0$ is the (trivial) solution. Therefore, the two given vectors are linearly independent.