Problem 15

Let

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} e^t \\ te^t \end{pmatrix}, \qquad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}.$$

Show that $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly dependent at each point in the interval $0 \le t \le 1$. Nevertheless, show that $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly independent on $0 \le t \le 1$.

Solution

At a particular value of t, say $t = t_0$, the first vector is a constant multiple of the second vector.

$$\mathbf{x}^{(1)}(t_0) = \begin{pmatrix} e^{t_0} \\ t_0 e^{t_0} \end{pmatrix} = e^{t_0} \begin{pmatrix} 1 \\ t_0 \end{pmatrix} = e^{t_0} \mathbf{x}^{(2)}(t_0)$$

Therefore, the two vectors are linearly dependent at each point in the interval $0 \le t \le 1$. The two vectors are linearly dependent on the interval if there exists a nontrivial solution to

$$c_1 \mathbf{x}^{(1)} + c_2 \mathbf{x}^{(2)} = \mathbf{0}$$

for c_1 and c_2. Rewrite this vector equation.

$$c_1 \begin{pmatrix} e^t \\ te^t \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The implied system of equations is as follows.

$$\begin{aligned} c_1 e^t + c_2 &= 0 \\ c_1 te^t + c_2 t &= 0 \end{aligned}$$

Dividing both sides of the second equation by t yields the first equation.

$$c_1 e^t + c_2 = 0$$

Only $c_1 = c_2 = 0$ can satisfy this equation for general t. Therefore, the two vectors are linearly independent on the interval $0 \le t \le 1$.