Problem 16

In each of Problems 16 through 25, find all eigenvalues and eigenvectors of the given matrix.

$$\begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$$

Solution

The aim is to solve the eigenvalue problem,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x},$$

where $\mathbf{A}$ is the given matrix. Bring $\lambda\mathbf{x}$ to the left side and combine the terms.

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0} \tag{1}$$

The eigenvalues satisfy

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

Evaluate the determinant and solve for λ.

$$\det \begin{pmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{pmatrix} = 0$$

$$(5-\lambda)(1-\lambda) + 3 = 0$$

$$\lambda^2 - 6\lambda + 8 = 0$$

$$(\lambda - 2)(\lambda - 4) = 0$$

$$\lambda = \{2, 4\}$$

Therefore, the eigenvalues are

$$\lambda_1 = 2 \quad \text{and} \quad \lambda_2 = 4.$$

Substitute λ_1 and λ_2 back into equation (1) to determine the corresponding eigenvectors, $\mathbf{x}_1$ and $\mathbf{x}_2$.

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{x}_1 = \mathbf{0}$$

$$\begin{pmatrix} 5-2 & -1 \\ 3 & 1-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$3x_1 - x_2 = 0$$

$$x_2 = 3x_1$$

$$\mathbf{x}_1 = \begin{pmatrix} x_1 \\ 3x_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{x}_2 = \mathbf{0}$$

$$\begin{pmatrix} 5-4 & -1 \\ 3 & 1-4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 - x_2 = 0$$

$$x_2 = x_1$$

$$\mathbf{x}_2 = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Note that x_1 is a free variable, or arbitrary constant.