Problem 17

In each of Problems 16 through 25, find all eigenvalues and eigenvectors of the given matrix.

$$\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$$

Solution

The aim is to solve the eigenvalue problem,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x},$$

where $\mathbf{A}$ is the given matrix. Bring $\lambda\mathbf{x}$ to the left side and combine the terms.

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0} \tag{1}$$

The eigenvalues satisfy

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

Evaluate the determinant and solve for λ.

$$\det\begin{pmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{pmatrix} = 0$$

$$(3-\lambda)(-1-\lambda) + 8 = 0$$

$$\lambda^2 - 2\lambda + 5 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$$

$$\lambda = \{1-2i, 1+2i\}$$

Therefore, the eigenvalues are

$$\lambda_1 = 1 - 2i \quad \text{and} \quad \lambda_2 = 1 + 2i.$$

Substitute λ_1 and λ_2 back into equation (1) to determine the corresponding eigenvectors, $\mathbf{x}_1$ and $\mathbf{x}_2$.

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{x}_1 = \mathbf{0}$$

$$\begin{pmatrix} 2+2i & -2 \\ 4 & -2+2i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} (2+2i)x_1 - 2x_2 &= 0 \\ 4x_1 + (-2+2i)x_2 &= 0 \end{aligned}\right\}$$

$$x_2 = (1+i)x_1$$

$$\mathbf{x}_1 = \begin{pmatrix} x_1 \\ (1+i)x_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 1+i \end{pmatrix}$$

$$(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{x}_2 = \mathbf{0}$$

$$\begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} (2-2i)x_1 - 2x_2 &= 0 \\ 4x_1 + (-2-2i)x_2 &= 0 \end{aligned}\right\}$$

$$x_2 = (1-i)x_1$$

$$\mathbf{x}_2 = \begin{pmatrix} x_1 \\ (1-i)x_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 1-i \end{pmatrix}$$

Note that x_1 is a free variable, or arbitrary constant.