Problem 18

In each of Problems 16 through 25, find all eigenvalues and eigenvectors of the given matrix.

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

Solution

The aim is to solve the eigenvalue problem,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x},$$

where $\mathbf{A}$ is the given matrix. Bring $\lambda\mathbf{x}$ to the left side and combine the terms.

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0} \tag{1}$$

The eigenvalues satisfy

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

Evaluate the determinant and solve for λ.

$$\det\begin{pmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{pmatrix} = 0$$

$$(-2-\lambda)(-2-\lambda) - 1 = 0$$

$$\lambda^2 + 4\lambda + 3 = 0$$

$$(\lambda+3)(\lambda+1) = 0$$

$$\lambda = \{-3, -1\}$$

Therefore, the eigenvalues are

$$\lambda_1 = -3 \quad \text{and} \quad \lambda_2 = -1.$$

Substitute λ_1 and λ_2 back into equation (1) to determine the corresponding eigenvectors, $\mathbf{x}_1$ and $\mathbf{x}_2$.

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{x}_1 = \mathbf{0}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} x_1 + x_2 &= 0 \\ x_1 + x_2 &= 0 \end{aligned}\right\}$$

$$x_2 = -x_1$$

$$\mathbf{x}_1 = \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{x}_2 = \mathbf{0}$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} -x_1 + x_2 &= 0 \\ x_1 - x_2 &= 0 \end{aligned}\right\}$$

$$x_2 = x_1$$

$$\mathbf{x}_2 = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Note that x_1 is a free variable, or arbitrary constant.