Problem 19

In each of Problems 16 through 25, find all eigenvalues and eigenvectors of the given matrix.

$$\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

Solution

The aim is to solve the eigenvalue problem,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x},$$

where $\mathbf{A}$ is the given matrix. Bring $\lambda\mathbf{x}$ to the left side and combine the terms.

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0} \tag{1}$$

The eigenvalues satisfy

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

Evaluate the determinant and solve for λ.

$$\det\begin{pmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(1-\lambda) + i^2 = 0$$

$$\lambda^2 - 2\lambda = 0$$

$$\lambda(\lambda - 2) = 0$$

$$\lambda = \{0, 2\}$$

Therefore, the eigenvalues are

$$\lambda_1 = 0 \quad \text{and} \quad \lambda_2 = 2.$$

Substitute λ_1 and λ_2 back into equation (1) to determine the corresponding eigenvectors, $\mathbf{x}_1$ and $\mathbf{x}_2$.

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{x}_1 = \mathbf{0}$$

$$\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} x_1 + ix_2 &= 0 \\ -ix_1 + x_2 &= 0 \end{aligned}\right\}$$

$$x_2 = ix_1$$

$$\mathbf{x}_1 = \begin{pmatrix} x_1 \\ ix_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{x}_2 = \mathbf{0}$$

$$\begin{pmatrix} -1 & i \\ -i & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} -x_1 + ix_2 &= 0 \\ -ix_1 - x_2 &= 0 \end{aligned}\right\}$$

$$x_2 = -ix_1$$

$$\mathbf{x}_2 = \begin{pmatrix} x_1 \\ -ix_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Note that x_1 is a free variable, or arbitrary constant.