Problem 2

In each of Problems 1 through 6, either solve the given system of equations, or else show that there is no solution.

\[
\begin{align*}
    x_1 + 2x_2 - x_3 &= 1 \\
    2x_1 + x_2 + x_3 &= 1 \\
    x_1 - x_2 + 2x_3 &= 1
\end{align*}
\]

Solution

Start by calculating the determinant of the coefficient matrix.

\[
\det \begin{pmatrix}
1 & 2 & -1 \\
2 & 1 & 1 \\
1 & -1 & 2
\end{pmatrix} = 1 \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} - 2 \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - 1 \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}
\]

\[
= 1(2 + 1) - 2(4 - 1) - 1(-2 - 1)
\]

\[
= 0
\]

Since it’s zero, either there is no solution to the system of equations or there are many.

\[
\begin{pmatrix}
1 & 2 & -1 & 1 \\
2 & 1 & 1 & 1 \\
1 & -1 & 2 & 1
\end{pmatrix}
\]

Multiply the first row by \(-2\) and add it to the second row.

\[
\begin{pmatrix}
1 & 2 & -1 & 1 \\
0 & -3 & 3 & -1 \\
1 & -1 & 2 & 1
\end{pmatrix}
\]

Multiply the first row by \(-1\) and add it to the third row.

\[
\begin{pmatrix}
1 & 2 & -1 & 1 \\
0 & -3 & 3 & -1 \\
0 & -3 & 3 & 0
\end{pmatrix}
\]

The last two rows of this augmented matrix imply that

\[
-3x_2 + 3x_3 = -1
\]
\[
-3x_2 + 3x_3 = 0,
\]
or

\[
0 = -1.
\]

This is a false statement, which means there’s no solution to the given system of equations.