

Problem 20

In each of Problems 16 through 25, find all eigenvalues and eigenvectors of the given matrix.

$$\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

Solution

The aim is to solve the eigenvalue problem,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x},$$

where \mathbf{A} is the given matrix. Bring $\lambda\mathbf{x}$ to the left side and combine the terms.

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0} \tag{1}$$

The eigenvalues satisfy

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

Evaluate the determinant and solve for λ .

$$\det \begin{pmatrix} 1 - \lambda & \sqrt{3} \\ \sqrt{3} & -1 - \lambda \end{pmatrix} = 0$$

$$(1 - \lambda)(-1 - \lambda) - 3 = 0$$

$$\lambda^2 - 4 = 0$$

$$(\lambda + 2)(\lambda - 2) = 0$$

$$\lambda = \{-2, 2\}$$

Therefore, the eigenvalues are

$$\lambda_1 = -2 \quad \text{and} \quad \lambda_2 = 2.$$

Substitute λ_1 and λ_2 back into equation (1) to determine the corresponding eigenvectors, \mathbf{x}_1 and \mathbf{x}_2 .

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{x}_1 = \mathbf{0}$$

$$\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left. \begin{aligned} 3x_1 + \sqrt{3}x_2 &= 0 \\ \sqrt{3}x_1 + x_2 &= 0 \end{aligned} \right\}$$

$$x_2 = -\sqrt{3}x_1$$

$$\mathbf{x}_1 = \begin{pmatrix} x_1 \\ -\sqrt{3}x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$$

$$(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{x}_2 = \mathbf{0}$$

$$\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left. \begin{aligned} -x_1 + \sqrt{3}x_2 &= 0 \\ \sqrt{3}x_1 - 3x_2 &= 0 \end{aligned} \right\}$$

$$x_2 = \frac{1}{\sqrt{3}}x_1$$

$$\mathbf{x}_2 = \begin{pmatrix} x_1 \\ \frac{1}{\sqrt{3}}x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

Note that x_1 is a free variable, or arbitrary constant.